Example

$$T : \mathbb{P}^1 \to \mathbb{R}^2 \qquad \{e_1, e_2\}$$

$$\underbrace{T(a+bx) = (a,b)}$$

$\{1, x\}$

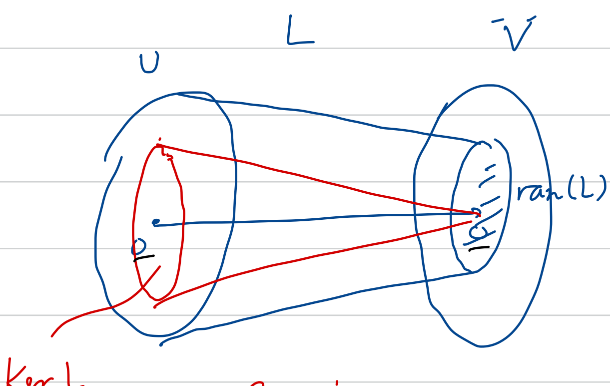
$$T : 1 \to e_1 = (1,0)$$
$$T : x \to e_2 = (0,1)$$

injective $\quad T(ax+b) = (0,0) \Rightarrow a=0, b=0 \Rightarrow a+bx=0$

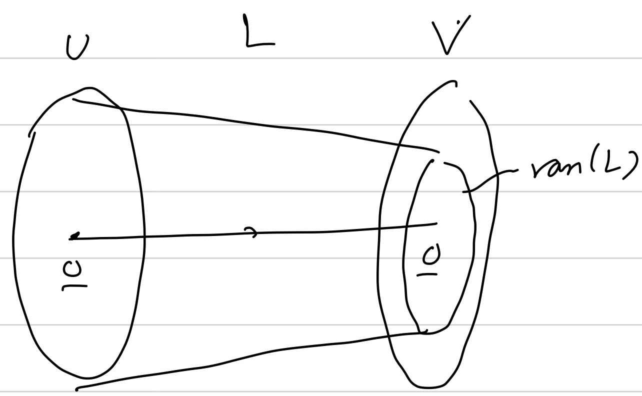
surjective ✓

$$\mathbb{P}^1 \simeq \mathbb{R}^2 \quad \text{isomorphic.}$$

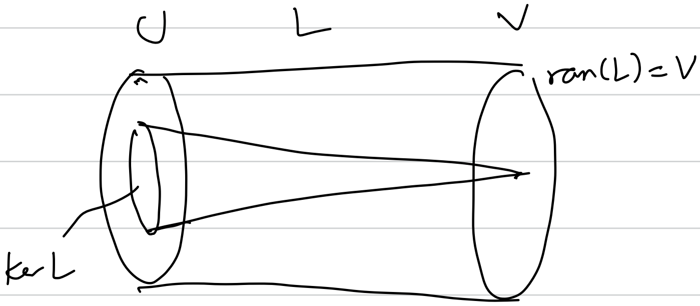
$$L : U \to V$$



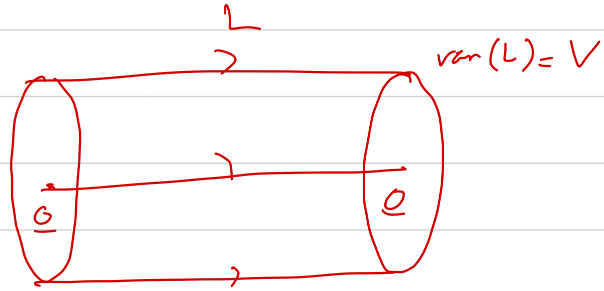
generic



injective



Surjective



ker(L) = {0}

Injectivity Criterion

Proposition $\;L : U \to V$ (linear) is injective **iff**

$$\ker L = \{\underline{0}\}$$

If L injects:

$$L(u) = 0 = L(0) \Rightarrow u = 0$$

If $\ker(L) = \{\underline{0}\}$

suppose $\quad L(u_1) = L(u_2)$

$$\Rightarrow L(u_1) - L(u_2) = 0$$

$$\Rightarrow L(u_1 - u_2) = 0$$

$$\Rightarrow u_1 - u_2 \in \ker(L) = \{\underline{0}\}$$

$$\Rightarrow u_1 - u_2 = \underline{0} \Rightarrow u_1 = u_2.$$

Matrix Representation

$$\left\{\begin{array}{l} L : U \xrightarrow[\text{isomorphism}]{} \mathbb{R}^{\dim U} \\ u \longmapsto [u]_B \to B \text{ a basis in } U. \end{array}\right.$$

Let $L : U \to V$ be linear.

Fix a basis (ordered) for U, say

$$B = (b_1, \dots, b_m) \qquad \boxed{m = \dim U}$$

let $\quad C = (c_1, \dots, c_n)$ be an ordered basis for V.
$\boxed{n = \dim V}$

$\{$ E : the standard basis for U

$\;\;$ F : " " " " V

Definition

matrix of L w.r.t the bases B, C

$$[L]_B^C := \big[[L(b_1)]_C \cdots\cdots [L(b_m)]_C \big] \in \mathbb{R}^{n \times m}$$

$$[L]_B := [L]_B^B \;, \qquad [L]_E =: [L]$$

E: standard basis

Proposition For all $u \in U$,

$$[L(u)]_C = [L]_B^C \, [u]_B \, .$$

Proof. Let $[u]_B = (\alpha_1, \ldots, \alpha_m)$. Then

$$L(u) = L(\alpha_1 b_1 + \cdots + \alpha_m b_m) = \alpha_1 L(b_1) + \cdots + \alpha_m L(b_m)$$

$$[L(u)]_C = \alpha_1 [L(b_1)]_C + \cdots + \alpha_m [L(b_m)]_C$$

$$= \big[[L(b_1)]_C \; \cdots \; [L(b_m)]_C \big] \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{bmatrix}$$

$$= [L]_B^C \, [u]_B$$

Diagram ($m = \dim U$, $n = \dim V$)

$$\begin{tikzcd}
\mathbb{R}^m \arrow[r, "L_A"] & \mathbb{R}^n \\
U \arrow[u, "u \mapsto [u]_B"] \arrow[r, "L"'] & V \arrow[u, "v \mapsto [v]_C"']
\end{tikzcd}$$

$$A := [L]_B^C$$

$$L_A(u) = Au$$

$$\begin{tikzcd}
{[u]_B} \arrow[r] & {[L]_B^C [u]_B = [L(u)]_C} \\
u \arrow[u] \arrow[r, "L"'] & L(u) \arrow[u]
\end{tikzcd}$$

<u>Corollary</u> :

$$[L]_B^C = \underbrace{(T_C^F)^{-1}}_{\substack{\text{natural} \\ \boxed{\text{transition}}}} \underbrace{[L]_B^F}_{\text{'natural'}}$$

<u>Proof</u>. For all $u \in U$:

$$\underbrace{(T_C^F)^{-1}}\underbrace{[L]_B^F} [u]_B = T_F^C [L(u)]_F = [L(u)]_C$$

$$[L]_B^C [u]_B = [L(u)]_C$$

$\boxed{\text{Ex}}$ Consider $\begin{cases} L : \mathbb{P}_2 \to \mathbb{P}_3 \\ L(p(x)) = p'(x) = \frac{d}{dx} p(x). \end{cases}$

Let $B = (x^2 - x + 1, 2x+3, 2x^2 - 3x + 4)$,

$C = (-x^2 + x + 1, -x^3 - 3x^2 + 2x + 3, x^3 - 3x^2 + 2, x^2 + 4x)$

be ordered bases for $\mathbb{P}_2$ and $\mathbb{P}_3$, respectively.

Find $[L]_B^C$.

Solution

The standard basis for $\mathbb{P}_2$ is $E = \{1, x, x^2\}$

" " " $\mathbb{P}_3$ is $F = \{1, x, x^2, x^3\}$

$$[L]_B^C = T_F^C \cdot [L]_B^F = (T_C^F)^{-1} \cdot [L]_B^F$$

$$= \Big[[-x^2+x+1]_F \;\; [-x^3-3x^2+2x+3]_F \;\; [x^3-3x^2+2]_F \;\; [x^2+4x]_F \Big]^{-1} \cdot$$

$$\Big[[L(\underbrace{x^2-x+1}_{2x-1})]_F \;\; [L(\underbrace{2x+3}_{2})]_F \;\; [L(\underbrace{2x^2-3x+4}_{4x-3})]_F \Big]$$

$$= \begin{bmatrix} 1 & 3 & 2 & 0 \\ 1 & 2 & 0 & 4 \\ -1 & -3 & -3 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} -1 & 2 & -3 \\ 2 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 26 & -5 & 20 & 8 \\ -5 & 1 & -4 & -2 \\ -5 & 1 & -4 & -1 \\ -4 & 1 & -3 & -1 \end{bmatrix}}_{(*)} \begin{bmatrix} -1 & 2 & -3 \\ 2 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -36 & 52 & -98 \\ 7 & -10 & 19 \\ 7 & -10 & 19 \\ 6 & -8 & 16 \end{bmatrix}_{4\times 3} .$$

* The main computational difficulty here is to find the inverse.

Example $L : \mathbb{R}^{2\times 2} \longrightarrow \mathbb{R}^{2\times 2}$

Define $L(X) = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} X$. Linear:

- $L(X+Y) = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}(X+Y) = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} X + \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} Y$
  $= L(X) + L(Y)$

$\cdot\ L(kX) = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}(kX) = k\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}X = kL(X)$

We find $[L]_B^C$ of L in the ordered bases

$$B = \left(\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 3 \\ -3 & 1 \end{bmatrix}, \begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & -4 \end{bmatrix} \right)$$

$$C = \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix} \right)$$

$[L]_B^C = (T_C^E)^{-1}\ [L]_B^E$, where

$$E = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

$$[L]_B^C = \begin{bmatrix} 0 & 2 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 1 & 3 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 3 & -7 & 9 & 2 \\ 0 & 5 & 5 & -6 \\ -1 & 3 & -2 & -1 \\ 1 & -1 & -1 & 4 \end{bmatrix}$$

$[L(b_2)]_F$

$$L(b_2) = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & 3 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -7 & 5 \\ 3 & -1 \end{bmatrix}$$

$$[L]_B^C = \begin{bmatrix} 0 & 5 & 5 & -6 \\ 8/5 & -4 & 28/5 & 2/5 \\ -2/5 & 1 & -11/5 & 6/5 \\ -1/3 & 1 & -2/3 & -1/3 \end{bmatrix}$$

* Notice that the matrix of L in the standard basis

$$[L] = [L]_E^E = \Big[\left[\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right]_E \; \left[\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right]_E \; \left[\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right]_E \; \left[\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right]_E \Big]$$

$$= \Big[\left[\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right]_E \; \left[\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right]_E \; \left[\begin{bmatrix} 2 & 0 \\ -1 & 0 \end{bmatrix} \right]_E \; \left[\begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix} \right]_E \Big]$$

$$= \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

---

subset vectorspace with an ordered basis B

<u>Definition</u>. Given $S \subseteq U$, we let

$$[S]_B := \{ [u]_B : u \in S \}$$

<u>Example</u> $S = \{1-x, 2+x\} \subseteq \mathbb{P}_1$, $E = \{1, x\}$

$[S]_E = \{(1,-1), (2,1)\}$.

Linear

$L : U \to V$

Proposition We have

$$[\mathrm{ran}(L)]_C = \mathrm{Col}\left([L]_B^C\right)$$

$$[\ker(L)]_B = \mathrm{Null}\left([L]_B^C\right)$$

Corollary :

$$\boxed{\begin{aligned} \mathrm{rank}(L) &= \mathrm{rank}\left([L]_B^C\right) \\ \mathrm{nullity}(L) &= \mathrm{nullity}\left([L]_B^C\right) \end{aligned}}$$

Example Let $L : \mathbb{R}^{2\times 2} \to \mathbb{P}_2$

$E = \left\{ \begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix} \right\}$

$F = \{1, x, x^2\}$

$$L(A) = 3\,\mathrm{tr}(A) + A(2,1)\,x^2$$

$$[L]_E^F = \left[\left[L\left(\begin{bmatrix}1&0\\0&0\end{bmatrix}\right)\right]_F \; \cdots \; \left[L\left(\begin{bmatrix}0&0\\0&1\end{bmatrix}\right)\right]_F \right] \qquad \underline{3\times 4}$$

$$= \left[[3 + 0x^2]_F \quad [3(0) + 0x^2]_F \quad [3(0) + 1\cdot x^2]_F \quad [3(1) + 0x^2]_F \right]$$

$$= \begin{bmatrix} 3 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

A basis for $\mathrm{Col}\left([L]_E^F\right) = \{(3,0,0), (0,0,1)\}$

$\Rightarrow$ A basis for $\mathrm{ran}(L)$ is $\{3, x^2\}$.

$\Rightarrow$ $\underline{\mathrm{rank}(L) = 2}$

A basis for $\underline{\text{Null}([L]_E^F)}$:

$$3x_1 + 3x_4 = 0 \quad \Rightarrow x_1 = -x_4$$
$$0 = 0 \qquad \underline{(x_1, x_2, x_3, x_4) = (-t, s, 0, t)}$$
$$x_3 = 0$$

$\Rightarrow$ a basis is $\{(-1,0,0,1), \underline{(0,1,0,0)}\}$

$\Rightarrow \left\{ \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$ is a basis for $\ker(L)$

$\Rightarrow \underline{\text{nullity}(L) = 2}$

Remark

$$\underline{\text{rank}(L) + \text{nullity}(L) = 2 + 2 = 4 = \dim \mathbb{R}^{2\times 2}}$$

Theorem (Dimension, Rank-Nullity)

for any linear transformation

$$\text{nullity}(L) + \text{rank}(L) = \dim(\underbrace{\text{dom}}_{\text{domain}}(L)).$$