Lecture 4

$$I \xrightarrow{e} E : \text{an elementary matrix}$$

(e: an elementary row operation)

Ex.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2 \xrightarrow{\overbrace{R_2 \leftarrow R_2 + 5R_1}^{e_1}} E_1 = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \xrightarrow{\overbrace{R_1 \leftarrow \frac{1}{2} R_1}^{e_2}} E_2 = \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \xrightarrow{\overbrace{R_1 \leftrightarrow R_2}^{e_3}} E_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Ex

$$A := \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}_{2\times 3}$$

$$E_1 A = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}_{2\times 2} \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}_{2\times 3}$$

$$= \begin{pmatrix} a & b & c \\ 5a+d & 5b+e & 5c+f \end{pmatrix}$$

$$A \xrightarrow{e_1} E_1 A$$

$$E_2 A = \begin{pmatrix} \frac{1}{2}a & \frac{1}{2}b & \frac{1}{2}c \\ d & e & f \end{pmatrix}$$

$$A \xrightarrow{e_2} E_2 A$$

$$A \xrightarrow{e_3} E_3 A = \begin{pmatrix} d & e & f \\ a & b & c \end{pmatrix}$$

Proposition If $I \xrightarrow{e} E$ then

$A \xrightarrow{e} EA$ for any A (with appropriate size)

If $I = I_n$, then $A \in \mathbb{R}^{n \times k}$.

Fact: Every elementary matrix in invertible.

If $I \xrightarrow{e} E$ and $I \xrightarrow{e^{-1}} F$, then

$\boxed{F = E^{-1}}$. Here e^{-1} means the inverse elementary

why?

row operation:

- $(R_i \leftrightarrow R_j)^{-1} = R_i \leftrightarrow R_j$
- $(R_i \leftarrow a R_i)^{-1} = R_i \leftarrow \frac{1}{a} R_i$

  $a \neq 0$
- $(R_i \leftarrow R_i + a R_j)^{-1} = R_i \leftarrow R_i - a R_j$

Proof: $EF = I$ and $FE = I$ because

$$I \xrightarrow{e} E \xrightarrow{e^{-1}} \underbrace{FE}_{\text{must be } I}$$

$\therefore$ $FE = I$

$I \xrightarrow{e^{-1}} F \xrightarrow{e} EF = I$. ∎

QED

Ex

$$E = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}$$

$$I \xrightarrow{R_2 \leftarrow R_2 + 5R_1 \, = e} E$$

$$I \xrightarrow{e^{-1} = R_2 \leftarrow R_2 - 5R_1} E^{-1} = \begin{pmatrix} 1 & 0 \\ -5 & 1 \end{pmatrix}$$

$$E \cdot E^{-1} = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -5 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \boxed{0} & 1 \end{pmatrix}$$

Ex

Find E^{-1} if

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \xrightarrow{e} \begin{pmatrix} 1 & -4 \\ 0 & 1 \end{pmatrix}$$

- $E = \begin{pmatrix} 1 & -4 \\ 0 & 1 \end{pmatrix} \longrightarrow E^{-1} = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$

$\boxed{e = R_1 \leftarrow R_1 - 4R_2} \Rightarrow e^{-1} = R_1 \leftarrow R_1 + 4R_2$

- $E = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} \longrightarrow E^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1/2 \end{pmatrix}$

$e = R_2 \leftarrow -2R_2$ $\qquad e^{-1} = R_2 \leftarrow -\frac{1}{2}R_2$

Algorithm for finding a matrix Inverse.

$$A \underbrace{\xrightarrow{e_1} \cdots \xrightarrow{e_2} \cdots \xrightarrow{e_3} \cdots \xrightarrow{e_n}}_{\text{Gauss - Jordan}} \underbrace{RRef(A)}_{\text{unique}}$$

$$A \xrightarrow{e_1} \cdots \xrightarrow{e_n} RRef(A)$$

reduced row echelon form

$$A \xrightarrow{e_1} E_1 A \xrightarrow{e_2} E_2 E_1 A \xrightarrow{e_3} E_3 E_2 E_1 A \to \cdots \xrightarrow{e_n}$$

$$E_n E_{n-1} \cdots E_1 A = RRef(A).$$

<u>Lemma</u>. If $A \xrightarrow{e_1} \cdots \xrightarrow{e_n} RRef(A)$ are the steps needed for Gauss-Jordan reduction of A, then $RRef(A) = \underline{E_n E_{n-1} \cdots E_1 A}$,

and $A = E_1^{-1} \cdots\cdots E_{n-1}^{-1} E_n^{-1} RRef(A)$.

where $I \xrightarrow{e_i} E_i$

<u>Proposition</u> If $\underline{RRef(A) = I}$, then A is invertible. Moreover, if

$$A \xrightarrow{e_1} \cdots \xrightarrow{e_n} RRef(A) = I, \text{ then}$$

$$I \xrightarrow{e_1} \cdots \xrightarrow{e_n} \underline{E_n E_{n-1} \cdots E_1} = A^{-1}$$

This can be written by flattening notation

$$[A \mid I] \xrightarrow{e_1} \cdots \xrightarrow{e_n} [I \mid A^{-1}]$$

Proof. Let $B = E_n E_{n-1} \cdots E_1$. Then

B is <u>invertible</u> (as a product of invertible matrices),

and moreover, $\underline{BA = I}$. But this

implies

$$\underline{AB} = I(AB) = (B^{-1}B)(AB)$$

$$= B^{-1}(BA)B = B^{-1} I B = B^{-1}B = I.$$

$\Rightarrow \underline{B = A^{-1}}.$

<u>Ex</u> What is X if

$$[A \mid C] \xrightarrow{e_1} \cdots \xrightarrow{e_n} [I \mid X] \; ?$$

$$A \xrightarrow{e_1} E_1 A \xrightarrow{e_2} E_2 E_1 A \to \cdots \to \overbrace{\boxed{E_n E_{n-1} \cdots E_1}}^{A^{-1}} A = \underline{\underline{I}}$$

$$C \longrightarrow \underbrace{E_n E_{n-1} \cdots E_1}_{A^{-1}} C = X$$

$$X = A^{-1} C$$

__Ex__ Find A^{-1} when $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

Gauss – Jordan

$$\left(\underbrace{\begin{matrix} 1 & 2 \\ 3 & 4 \end{matrix}}_{A} \middle| \underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{I}\right) \xrightarrow{R_2 \leftarrow R_2 - 3R_1} \left(\begin{matrix} 1 & 2 \\ 0 & -2 \end{matrix} \middle| \begin{matrix} 1 & 0 \\ -3 & 1 \end{matrix}\right)$$

$$\xrightarrow{R_1 \leftarrow R_1 + R_2} \left(\begin{matrix} 1 & 0 \\ 0 & -2 \end{matrix} \middle| \begin{matrix} -2 & 1 \\ -3 & 1 \end{matrix}\right)$$

$$\xrightarrow{R_2 \leftarrow -\frac{1}{2} R_2} \left(\underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{I} \middle| \underbrace{\begin{matrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{matrix}}_{A^{-1}}\right).$$

check $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

__Lemma__ A square matrix with a zero row is __not__ invertible.

__Proof.__ If $A = \begin{pmatrix} & ? & \\ 0 - & 0 & - 0 \\ & ? & \end{pmatrix} \in \mathbb{R}^{n \times n}$

(ith row is $\underline{0}$)

$\Rightarrow$ AB will have ith row $= 0$ as well, for any B, meaning $AB \neq I$, for any B. ∎

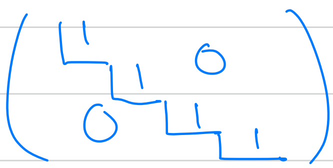
Prop. $A \in \mathbb{R}^{n\times n}$ is invertible <u>iff</u> <u>$\mathrm{RRef}(A) = I_n$</u>.

<u>proof</u> ("$\Rightarrow$" needs proof now).

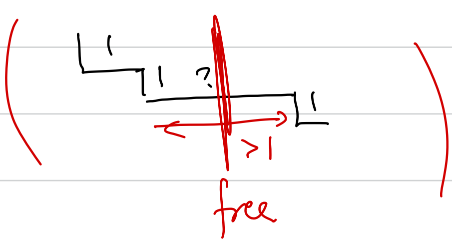
Suppose

$$R := \mathrm{RRef}(A) = E_n E_{n-1} \cdots E_1 A .$$

- If R <u>has no free columns</u>, the number of leading 0's always increase by 1, therefore <u>$R = I$</u>.

$R =$ 

- If R has at least one free column, then the number of leading 0's must increase by more than 1 in at least one of the rows.

$R =$ 

$n\times n$ square

$\Rightarrow$ R has a <u>zero row</u> (at the bottom) at least.

$\Rightarrow$ R is not invertible

$$R = E_n E_{n-1} \cdots E_1 A$$

$\Rightarrow$ $\underline{A \text{ is not invertible}}$. $\square$

<u>Corollary</u>: If A is noninvertible, then $RRef(A)$ is of the form

$$\begin{pmatrix} & ? & \\ 0 & \cdots & 0 \end{pmatrix}.$$

<u>Corollary</u>. An <u>invertible</u> matrix is a product of elementary matrices. In fact, if

$$A \xrightarrow{e_1} E_1 A \to \cdots \xrightarrow{e_n} E_n E_{n-1} \cdots E_1 A = I$$

are steps in Gauss-Jordan reduction, then

$$A = (E_n \cdots E_1)^{-1} = E_1^{-1} \cdots E_{n-1}^{-1}.$$

<u>Ex</u> Write $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ as a product of elementary matrices.

<u>Recall</u>

$$A \xrightarrow[R_2 \leftarrow R_2 - 3R_1]{e_1} E_1 A \xrightarrow[R_1 \leftarrow R_1 + R_2]{e_2} E_2 E_1 A \xrightarrow[R_2 \leftarrow -\frac{1}{2} R_2]{e_3} E_3 E_2 E_1 A = I$$

$$A = E_1^{-1} E_2^{-1} E_3^{-1} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$$

$e_1^{-1} = R_2 \leftarrow R_2 + 3R_1$ $\quad$ $R_1 \leftarrow R_1 - R_2$ $\quad$ $R_2 \leftarrow -2R_2$

Determinant

Invertibility

$[a]$ invertible iff $a \neq 0$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow[R_2 \leftarrow aR_1]{R_1 \leftarrow cR_1} \begin{bmatrix} ac & bc \\ ca & da \end{bmatrix}$$

assume $a, c \neq 0$

$$\xrightarrow{R_2 \leftarrow R_2 - R_1} \begin{bmatrix} ac & bc \\ 0 & ad - bc \end{bmatrix}$$

Singular : $ad - bc = 0$

define $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$

notice $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a \det[d] - b \det[c]$

$\det[a] = a$

Definition The determinant of a matrix $A \in \mathbb{R}^{n \times n}$ is defined recursively as a cofactor expansion along the i^{th} row

$$\det(A) = \sum_{j=1}^{n} \boxed{(-1)^{i+j}} A(i,j) \det(A_{ij})$$

where $A_{ij} \in \mathbb{R}^{(n-1)\times(n-1)}$ is obtained deleting the ith row and jth column of A

This definition is independent of which row we are expanding about. (Laplace)

$(-1)^{i+j}$: chessboard sign pattern

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$