$L : V \to V$, If $v \neq \underline{0}$ such that $L(v) = \lambda v$ for some real number λ.

(λ, v) : eigenpair

λ ↓ eigenvalue for L

v → a corresponding eigenvector

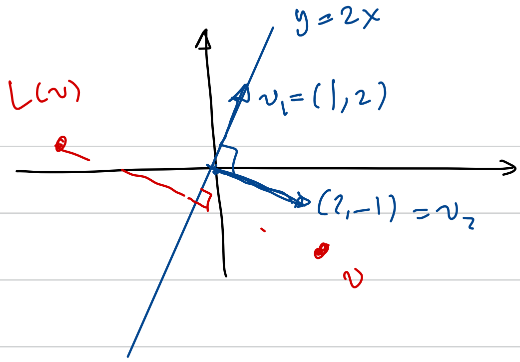
<u>Proposition</u> $v \neq \underline{0}$ is an eigenvector of $L : V \to V$ with eigenvalue λ $\iff$ $v \in \ker(L - \lambda \underbrace{I}_{\text{identity}})$

<u>Proof</u>

$$L(v) = \lambda v \iff L(v) - \lambda v = \underline{0}$$
$$\iff (L - \lambda I)v = \underline{0}$$
$$\iff v \in \ker(L - \lambda I).$$

<u>Def'n</u> $E_\lambda := \ker(L - \lambda I)$, which is a subspace of V is called the eigenspace of L corresponding to λ.

$\boxed{\text{Ex}}$ (Reflection about $y = 2x$ in $\mathbb{R}^2$)



$$\begin{cases} L(v_1) = v_1 & (\lambda_1 = 1) \\ v_1 \in E_1 = \ker(L - I) \end{cases}$$

$$\begin{cases} L(v_2) = -v_2 & (\lambda_2 = -1) \\ v_2 \in E_{-1} = \ker(L + I) \end{cases}$$

$E_1 = \operatorname{span}\{(1,2)\} \longrightarrow$ geometric multiplicity $= \dim E_1 = 1$

$E_{-1} = \operatorname{span}\{(2,-1)\}$ $\qquad \dim E_{-1} = 1$

$*$ geometric multiplicity of an eigenvalue λ of L $= \dim E_\lambda = \dim \ker(L - \lambda I)$

$= \underline{\underline{\text{nullity of } L - \lambda I}}$

<u>Def'n</u> If $A \in \mathbb{R}^{n\times n}$, $\quad Ax = \lambda x \quad , \quad x \neq \underline{0} \in \mathbb{R}^n$

$$L(x) = L_A(x) := Ax \qquad \left(\begin{array}{l} \text{eigenvalue} : \lambda \\ \text{eigenvector} : x \neq 0 \end{array} \right)$$

<u>Proposition</u> The vector v is an eigenvector of L with an eigenvalue $\underline{\underline{\lambda}}$ $\iff$ $[v]_B$ is an eigenvector of $[L]_B = [L]_B^B$ with eigenvalue $\underline{\underline{\lambda}}$.

$$L(v) = \lambda v \implies [L(v)]_B = [\lambda v]_B$$

$$[L]_B [v]_B = \lambda [v]_B$$

Proposition A scalar λ is an eigenvalue of $\overset{\in \mathbb{R}^{n\times n}}{\overset{\downarrow}{A}} \iff$

$$\boxed{\det(A-\lambda I) = 0}$$

Proof. λ is eigenvalue of $A \iff$ $\underline{\underline{\text{Null}(A-\lambda I)}}$ is $\underline{\underline{\text{nontrivial}}}$ (i.e.: $(A-\lambda I)x = \underline{0}$ has nonzero solution x). $\iff$ $\underline{\underline{\det(A-\lambda I) = 0}}$

Def'n: The characteristic polynomial of $A \in \mathbb{R}^{n\times n}$ is the polynomial

$$\text{charpoly}(A,\lambda) := \det(A-\lambda I).$$

$$\deg \text{Charpoly}(A,\lambda) = n \qquad A \in \mathbb{R}^{n\times n}$$

Notice: Each eigenvalue of A is a root of Charpoly(A,λ).

Def'n The algebraic multiplicity of an eigenvalue λ is the multiplicity of λ as a root of Charpoly(A,λ).

$$p(\lambda) = (\lambda-1)^3(\lambda+3)^2$$

$\lambda = 1$ has $\underline{\text{multiplicity}}$ $\underline{3}$

$\lambda = -3$ " $\underline{\quad " \quad}$ $\underline{2}$

Def'n The Spectrum of a square matrix or a linear operator is the set of eigenvalues.

$$\sigma(A) \quad \text{or} \quad \sigma(L)$$

It is clear $\sigma(L) = \sigma([L]_B)$

Example $A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$. Find the spectrum of A.

$$\text{Charpoly}(A,\lambda) = \det(A - \lambda I) = \det\left(\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$$

$$= \det\begin{pmatrix} 1-\lambda & 2 \\ 0 & 3-\lambda \end{pmatrix} = (1-\lambda)(3-\lambda) - 0\cdot 2 = (1-\lambda)(3-\lambda)$$

$\Rightarrow$ eigenvalues of A are $\lambda = 1, 3$, $\Rightarrow \sigma(A) = \{1, 3\}$.

$$\begin{cases} \text{algebraic multiplicity of } (\lambda = 1) = 1 \\ \quad " \qquad " \qquad " \quad (\lambda = 3) = 1 \end{cases}$$

Proposition If $A \sim B$ then $\text{Charpoly}(A,\lambda) = \text{Charpoly}(B,\lambda)$.

$A = P^{-1} B P$

$$\text{Charpoly}(A,\lambda) = \text{Char}(P^{-1}BP, \lambda) = \det(P^{-1}BP - \lambda \overset{P^{-1}IP}{I})$$

$$= \det\left(P^{-1}(B - \lambda I)P\right) = \underbrace{\det(P^{-1})} \det(B - \lambda I) \underbrace{\det(P)}$$

reciprocal

$= \det(B - \lambda I) = \text{charpoly}(B, \lambda).$

<u>Def'n</u> The characteristic polynomial of $L : V \to V$ is $\text{charpoly}(L, \lambda) := \det\left([L]_B - \lambda I\right)$ where B is any ordered basis for V.

$$[L]_C \sim [L]_B \qquad B, C : \text{different ordered bases of } V.$$

<u>Example</u> $0 \in \mathbb{R}$ is an eigenvalue of A (or L) if $\ker(A - 0I) = \ker(A)$ is nontrivial. meaning A is not invertible (singular).

0 is not eigenvalue of A. $\iff$ A is invertible.

<u>Diagonlization</u>

An ordered basis $\underline{B = (b_1, \ldots, b_n)}$ consisting of eigenvectors of L leads to the equation

$$L(b_j) = \lambda_j b_j \qquad j = 1, \ldots, n$$

$$[L]_B = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ \vdots & & \ddots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$ is a diagonal matrix

eigenvalues all appear on the diagonal.

<u>Def'n</u> A linear operator $L: V \to V$ or a matrix A is diagonalizable if there exists a basis consisting of eigenvectors.

Algorithm To diagonalize a square matrix A:

1) Find the characteristic polynomial.
2) Find the spectrum as the roots of the characteristic polynomial
3) For each eigenvalue λ, find a basis for the

$$\text{Null}(A - \lambda I) = E_\lambda$$

4) If the algebraic and geometric multiplicities for any λ do not match, then A is not diagonalizable.

5) Build a matrix P using the bases for the eigenspaces as columns.

6) Build a diagonal matrix D from the eigenvalues of the columns of P.

7) $D = P^{-1} A P$

$\boxed{Ex}$ We diagonalize $A = \begin{pmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{pmatrix}$

$$\text{Charpoly}(A,\lambda) = \det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -1 & -2 \\ 2 & 0-\lambda & -2 \\ 2 & -1 & -1-\lambda \end{vmatrix}$$

$$= -\lambda^3 + 2\lambda^2 - \lambda = -\lambda(\lambda^2 - 2\lambda + 1) = -\lambda(\lambda-1)^2$$

$\sigma(A) = \{0, 1\}$

alg. multiplicity: 1 2

$$\begin{array}{c} \begin{array}{ccc} & & \text{free} \end{array} \\ A \xrightarrow[\text{reduction}]{\text{Row}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \end{array}$$

$E_0, E_1 = ?$

$x_1 - x_3 = 0$
$x_2 - x_3 = 0$

* $E_0 = \text{Null}(A - 0I) = \text{Null}(A) = \text{Span}\{(1,1,1)\}$

$\text{nullity}(A - 0I) = 1$ (# free columns in echelon form)

$\Rightarrow$ geom. multip. of $\lambda = 0$ is 1. ✓

also algebraic

* $E_1 = \text{Null}(A - I) = \text{Null}\begin{pmatrix} 2 & -1 & -2 \\ 2 & -1 & -2 \\ 2 & -1 & -2 \end{pmatrix} = \text{Null}\begin{pmatrix} 2 & -1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

leading free

$2x_1 - x_2 - 2x_3 = 0$

free

$2x_1 = x_2 + 2x_3 \Rightarrow x_1 = \frac{1}{2}x_2 + x_3$

$(x_1, x_2, x_3) = (\frac{1}{2}x_2 + x_3, x_2, x_3) = x_2(\frac{1}{2}, 1, 0) + x_3(1, 0, 1)$

$$\mathrm{Null}(A - I) = \mathrm{span}\left\{ \left(\tfrac{1}{2}, 1, 0\right), (1,0,1) \right\}.$$

geom. multip. of $\lambda = 1$ is $\underline{\underline{2}}$ = algebraic multip..

$$B = \{(1,1,1), (1/2, 1, 0), (1,0,1)\}$$

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [L_A]_B^B = T_E^B \underbrace{[L_A]_E^E}_{A} \underbrace{T_B^E}_{P}$$

$$T_E^B = (T_B^E)^{-1} = P^{-1}$$

$$= \begin{bmatrix} 1 & 1/2 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}^{-1} \underbrace{\begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{bmatrix}} \begin{bmatrix} 1 & 1/2 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$\boxed{\text{Ex}}$ $A^{50} = ?$

$$D = P^{-1} A P \implies A = P D P^{-1}$$

$$\implies A^2 = (PDP^{-1})(PDP^{-1}) = PD^2P^{-1}$$

$$A^3 = (PDP^{-1})(PDP^{-1})(PDP^{-1}) = PD^3P^{-1}$$

$$\vdots$$

$$\underbrace{A^k = PD^kP^{-1}} \qquad D^k = \begin{pmatrix} \lambda_1^k & & 0 \\ & \lambda_2^k & \\ 0 & & \ddots \; \lambda_n^k \end{pmatrix}$$

$$A^{50} = P \underset{\substack{\downarrow \\ D \text{ b/c } \underline{\lambda = 0,1}}}{D^{50}} P^{-1} = PDP^{-1} = A.$$

Example Who's Cool? Andy, Bob, Caren, Debra

Who's the coolest Person?

- Andy has a crush on Caren.
- Bob likes Andy and Caren.
- Caren thinks only Debra is cool.
- Debra is a secret admirer of Bob.

$$
\begin{tikzcd}
A \arrow[r] & C \arrow[d] \\
B \arrow[u] \arrow[ur] & D \arrow[l]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
x_1 \arrow[r, "1/2"] \arrow[d, shift left, "1/2"] & x_3 \arrow[d, "1"] \\
x_2 \arrow[u, shift left, "1/2"] \arrow[ur, "1/2"'] & x_4 \arrow[l, "1"]
\end{tikzcd}
$$

Let $x_i \in R$ be coolness factor of the ith person

$$
\begin{aligned}
0.5x_2 &= x_1 \\
0.5x_1 + x_4 &= x_2 \\
0.5x_1 + 0.5x_2 &= x_3 \\
x_3 &= x_4
\end{aligned}
\qquad
\underbrace{\begin{bmatrix} 0 & 0.5 & 0 & 0 \\ 0.5 & 0 & 0 & 1 \\ 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{Ax = x}
$$

$\lambda = 1$

$$\text{Charpoly}(A, \lambda) = \lambda^4 - \frac{1}{4}\lambda^2 - \frac{1}{2}\lambda - \frac{1}{4}$$

($\lambda = 1$) is a root

$$E_1 = \text{Span}\{(2, 4, 3, 3)\}$$

Bob is the coolest!

Normed Space "measure length of vectors"

$\|\cdot\| : V \longrightarrow \mathbb{R}$ "norm" is function

Properties

$\forall v \in V$

(1) $\|v\| \geq 0$, $\|v\| = 0 \Leftrightarrow v = \underline{0}$

(2) $\|cv\| = |c|\,\|v\|$ $\forall c \in \mathbb{R}, \forall v \in V$

(3) $\|u+v\| \leq \|u\| + \|v\|$ $\forall u,v \in V$

A normed space : $(\underbrace{V}_{\text{vector space}}, \underbrace{\|\cdot\|}_{\text{norm on } V.})$

$\underline{\mathbb{R}^n}$ p-norm (ℓ^p-norm) $p \geq 1$, $p = \infty$

$$\|v\|_p = \left(|v_1|^p + \cdots |v_n|^p\right)^{1/p} \qquad p \geq 1$$

$v = (v_1, \ldots, v_n)$

$$\|v\|_\infty = \max\{|v_1|, \ldots, |v_n|\}$$

ℓ^2 - norm : $\|v\|_2 = \left(\sum_{j=1}^{n} |v_j|^2\right)^{1/2}$ Euclidean norm ℓ^2-norm.

$$\|v\|_\infty \leq \|v\|_2 \leq \|v\|_1$$

$V = C([0,1])$ continuous function $f: [0,1] \to \mathbb{R}$

$$\|f\|_p := \left(\int_a^b |f(x)|^p \, dx \right)^{\frac{1}{p}}$$

L^p-norm

$\underline{p \geq 1}$

$$\|f\|_\infty := \max_{a \leq x \leq b} |f(x)|$$