Lecture 3

Flattening construction: build larger matrices out of smaller ones

$A := [1 \quad 2]$

$B := \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$

$C := \begin{bmatrix} 7 & 8 \\ 9 & 10 \end{bmatrix}$

$[B \quad C] = \begin{bmatrix} 3 & 4 & 7 & 8 \\ 5 & 6 & 9 & 10 \end{bmatrix}$

$[B \quad A^T] = \begin{bmatrix} 3 & 4 & 1 \\ 5 & 6 & 2 \end{bmatrix}$: augmented

$[\pi \quad A] = [\pi \quad 1 \quad 2]$

$e_1, \dots, e_n$: unit vectors in $\mathbb{R}^n$,

$I_n = [e_1 \; e_2 \; \cdots \; e_n]$: the identity matrix of size n.

Basic matrix operations

$A \in \mathbb{R}^{m \times n}$, define A^T (the transpose of A) $\in \mathbb{R}^{n \times m}$

$$A^T(i,j) := A(j,i)$$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$(1,2)^T = [1 \quad 2]$ $\qquad I^T = I$

Symmetric matrix: (square such that) $A^T = A$.

$A(i,j) = A(j,i)$

$A = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$ is symmetric

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ $\quad A = A^T \Leftrightarrow b = c$

$\begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix}$ is symmetric.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Properties $\quad (A^T)^T = A \ , \ (A+B)^T = A^T + B^T$

$$(cA)^T = cA^T$$

Definition $\quad$ If $A \in \mathbb{R}^{n \times n}$, then trace of A is a real number

$$tr(A) := \sum_{k=1}^{n} A(k,k)$$

$tr \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} = 1 + 3 = 4 \qquad tr(I_n) = n$

$$tr(A^T) = tr(A)$$

$\left(b/c \ \sum_{k=1}^{n} A^T(k,k) = \sum_{k=1}^{n} A(k,k) \right)$

Linear Combination

A linear combination of $\underline{\text{objects}}$ $v_1, \dots, v_n$ is of the form

$\downarrow$ different things

$$\sum_{i=1}^{n} c_i v_i = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$$

where $c_i \in \mathbb{R}$.

objects could be, for example:

matrices, n-tuples, functions, polynomials, sequences

Ex

Write $p(x,y) = 2x^2 + 2y^2$ as a linear combination of

$q(x,y) = (x-y)^2$ and $r(x,y) = xy$

$$2x^2 + 2y^2 \stackrel{?}{=} c_1 (x-y)^2 + c_2 xy \qquad \text{for some } c_1, c_2 \in \mathbb{R}$$

$c_1 \to 2$, $c_2 \to 4$

Linear Combination of Matrices

Let $\underline{A, B \in \mathbb{R}^{m\times n}}$. Define $A+B$, $A-B$, cA, $-A \in \mathbb{R}^{m\times n}$

$\underline{\text{componentwise}}$

For example:

$$(A-B)(i,j) := A(i,j) - B(i,j)$$

Notice

$$A - B = A + (-B) \quad , \quad -B = (-1)B .$$

Properties

$$A + B = B + A \qquad \text{Commutative}$$

$c, d \in \mathbb{R}$

$A, B, C \in \mathbb{R}^{n \times n}$

$$A + (B + C) = (A + B) + C \qquad \text{associative}$$

$$c(A + B) = cA + cB \qquad \text{distributive}$$

$$(c + d)A = cA + dA \qquad //$$

$$(cd)A = c(dA) \qquad \text{associative}$$

Matrix Product

$A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{n \times k}$, define

$$(AB)(\underline{i}, \underline{j}) := \text{" take the dot product between the } i^{th} \text{ row of } A \text{ and } j^{th} \text{ column of } B \text{"}$$

$AB \in \mathbb{R}^{m \times k}$

$$= \sum_{l=1}^{n} A(\underline{i}, l)\, B(l, \underline{j})$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2 \times 2} \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}_{2 \times 2} = \begin{bmatrix} 1(2) + 2(0) & 1(-1) + 2(3) \\ 3(2) + 4(0) & 3(-1) + 4(3) \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 6 & 9 \end{bmatrix}$$

size of the product

$$[a \;\; b]\,[x \;\; y]^T = [a \;\; b] \begin{bmatrix} x \\ y \end{bmatrix} = [ax + by]_{1 \times 1}$$

Properties

$\boxed{A, B, C \text{ matrices of "appropriate size"}}$

$A(BC) = (AB)C$ associative

$A(B+C) = AB + AC$

$(A+B)C = AC + BC$

$c(AB) = (cA)B = A(cB)$

$IA = A, \quad AI = A$

Remark If AB is defined, BA might not be defined

① $A \in \mathbb{R}^{2\times 3}$, $B \in \mathbb{R}^{3\times 3}$ $\Rightarrow$ $AB \in \mathbb{R}^{2\times 3}$

BA undefined.

② A, B, AB & BA are defined if

$A \in \mathbb{R}^{m\times n}$ $B \in \mathbb{R}^{n\times m}$ $\Rightarrow \begin{cases} AB \in \mathbb{R}^{m\times m} \\ BA \in \mathbb{R}^{n\times n} \end{cases}$

③ Even $A, B \in \mathbb{R}^{n\times n}$, $AB \neq BA$ in general

$$\begin{pmatrix}1&2\\3&4\end{pmatrix}\begin{pmatrix}a&b\\c&d\end{pmatrix}=\begin{pmatrix}a+2c & b+2d\\3a+4c&3b+4d\end{pmatrix}$$

$$\neq \begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}1&2\\3&4\end{pmatrix}=\begin{pmatrix}a+3b&2a+4b\\c+3d&2c+4d\end{pmatrix}$$

They are equal <u>iff</u>

$$\underbrace{\begin{cases} a+2c = a+3b \\ b+2d = 2a+4b \\ c+3d = 3a+4c \\ 3b+4d = 2c+4d \end{cases}}_{\text{homogeneous}}$$

$$A' = \begin{pmatrix}1&2\\3&4\end{pmatrix}' = \text{commutant of } A = \{ B \mid AB = BA \}$$

How many ways can we get there?

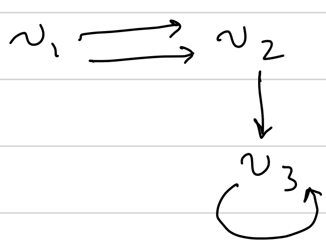
<u>Definition</u>

A directed multigraph :

$$\begin{cases} \text{Vertices } \{v_1, \ldots, v_n\} \\ \text{Edges : paths from } v_i \text{ to } v_j \end{cases}$$

Adjacency matrix :

$A(i,j) :=$ # of arrows from v_i to v_j



$$\Downarrow$$

$$A \in \mathbb{R}^{3\times 3}$$

$$A = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

notice $A^2 = AA = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ there are two paths of length $\underline{\underline{2}}$ from v_1 to v_3

Proposition

$A^n(i,j) = \#$ walks from v_i to v_j in $\underline{n \text{ steps}}$.

(Try to prove this for $n = 2$.)

Matrix form of system of Equations

$$\underbrace{\begin{aligned} x + 2y &= 5 \\ 3x - y &= 1 \end{aligned}} \Longrightarrow \underbrace{\begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

$A =$ Coefficient matrix

Note that

$$\underbrace{x \begin{bmatrix} 1 \\ 3 \end{bmatrix} + y \begin{bmatrix} 2 \\ -1 \end{bmatrix}} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

a specific linear combination of the columns of A

meaning the coefficients vector is a linear combination of the columns of A

* If $A = [C_1 \; C_2 \cdots C_n]$ with columns $C_1, \ldots, C_n$

then $$A \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \underbrace{a_1 C_1 + \cdots + a_n C_n = \sum_{j=1}^{n} a_j C_j}_{\text{a linear combination of } C_1, \ldots, C_n.}$$

<u>Ex</u> Can we write $(1,2)$ as a linear combination of $(1,1)$ and $(-1,2)$?

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

<u>Augmented</u>: $$\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 1 & 2 & 2 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - R_1} \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 3 & 1 \end{array}\right]$$

$$3b = 1 \Rightarrow b = 1/3 \xrightarrow{\text{backward}} a - b = 1 \Rightarrow a = 4/3$$

$$\underline{(1,2) = 4/3\,(1,1) + 1/3\,(-1,2)}$$

Inverse Matrix

A, B square matrices

If $AB = BA = I$, we write

$A = B^{-1}$ (or same as $B = A^{-1}$)

If so,

A is called invertible or regular

Otherwise, non-invertible or singular.

Exercise Find the inverse of $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$.

Let $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and demand

$$AB = BA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$(A^{-1})^{-1} = A$$

$$\boxed{(AB)^{-1} = B^{-1}A^{-1}}$$

if A, B are invertible matrices

Proof.

1) the inverse of an invertible matrix is unique

$AB = BA = I$ and $AC = CA = I$

$$B = BI = B(AC) = (BA)C = IC = C.$$

(therefore, write $B = C = A^{-1}$)

2) $$(AB)B^{-1}A^{-1} = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

and

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$$

Proposition If we have a system of n equations in n variables

$$Ax = b$$

A: coeff. matrix $\in \mathbb{R}^{n \times n}$

x, b: in $\mathbb{R}^{n \times 1}$

and if A is invertible, then

$$x = A^{-1} b \qquad \text{(uniquely)}.$$

$$A^{-1}(Ax = b) \implies (A^{-1}A)x = A^{-1}b$$

$$I x = A^{-1} b$$

$$x = A^{-1} b.$$

Elementary matrices :

A matrix obtained from the identity matrix using an elementary row operation.

Notation : $I \xrightarrow{e} \underline{E}$

2x2

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_2 \leftarrow 3R_2} E_2 = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftarrow R_1 - 2R_2} E_3 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

Write all 2x2 elementary matrices .