If $A \in \mathbb{R}^{m \times n}$, then

$$\text{rank}(A) = \dim \overbrace{\text{Col}(A)}^{\subset \mathbb{R}^m} \le m$$

$$\text{nullity}(A) = \dim \underbrace{\text{Null}(A)}_{\subset \mathbb{R}^n} \le n$$

Rank-Nullity Theorem

$$\underbrace{\text{rank}(A)}_{\#\text{ leading columns in } R} + \underbrace{\text{nullity}(A)}_{\#\text{ free columns in } R} = \#\text{ columns of } A = n$$

R: a row-echelon form for A =

4 Linear Transformations

Def'n U, V : vector spaces

function $L : U \to V$ is a linear transformation

if

1) $L(u+v) = L(u) + L(v)$, for all $u, v \in U$

(additive)

2) $L(\alpha v) = \alpha L(v)$, for all $\alpha \in \mathbb{R}$, $v \in V$

(multiplicative)

If $U = V$, we call L a linear operator.

Examples ① $L: U \to V$ $L(u) = \underline{0} \in V, \ \forall u \in U$

'zero map'

② $L: U \to U$ identity map

$L(u) = u$

(or $L(u) = k u$, $k \in \mathbb{R}$ fixed.)

③ $L: \mathbb{R}^2 \to \mathbb{R}^3$ $L(a,b) = (b-a, 0, 3a)$

$$L(a,b) = L(a\underbrace{e_1}_{(1,0)} + b\underbrace{e_2}_{(0,1)}) = L(ae_1) + L(be_2)$$

$$= a\underbrace{L(e_1)}_{(\alpha_1, \alpha_2, \alpha_3)} + b\underbrace{L(e_2)}_{(\beta_1, \beta_2, \beta_3)}$$

$$= a(\alpha_1, \alpha_2, \alpha_3) + b(\beta_1, \beta_2, \beta_3)$$

$$= (\alpha_1 a + \beta_1 b, \ \alpha_2 a + \beta_2 b, \ \alpha_3 a + \beta_3 b)$$

For example, $L(a,b) = (b-a, 0, 3a)$

$$= a\underbrace{(-1, 0, 3)}_{L(e_1)} + b\underbrace{(1, 0, 0)}_{L(e_2)}$$

④ $L : \mathbb{R}^{m\times n} \to \mathbb{R}^{n\times m}$

$L(A) = A^T$

$\begin{cases} (A+B)^T = A^T + B^T \\ (kA)^T = kA^T \end{cases}$

⑤ $L : \mathbb{R}^{n\times n} \to \mathbb{R}$

$L([a_{ij}]_{n\times n}) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}$

or $L(A) = \underline{tr(A)}$

$\underline{n=2}$ $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

tr $\downarrow$ $\downarrow$ $\downarrow$ $\downarrow$

$1 \quad 0 \quad 0 \quad 1$

⑥ $L : \mathbb{P}_n \to \mathbb{R}_n$

$L(p(x)) = \frac{d}{dx} p(x)$

⑦ $L : \mathbb{P}_n \to \mathbb{P}_{n+1}$

$L(p(x)) = \int_0^x p(t)\,dt$, or $L(p(x)) = x\,p(x)$

$\underbrace{x(p(x)+q(x))}_{L(p+q)} = \underbrace{xp(x)}_{L(p)} + \underbrace{xq(x)}_{L(q)}$.

$L(kp) = x(kp) = k(xp)$

⑧ $L : \mathbb{P}_n \to \mathbb{R}$

$$L(p(x)) = p(2) - 3p(5)$$

$$\begin{aligned} L(p(x)+q(x)) &= (p+q)(2) - 3(p+q)(5) \\ &= p(2)+q(2) - 3p(5) - 3q(5) \\ &= (p(2) - 3p(5)) + (q(2) - 3q(5)) \\ &= L(p) + L(q). \end{aligned}$$

$E = \{1, x, x^2, \dots, x^n\}$ standard basis for $\mathbb{P}_n$.

$$\begin{aligned} L(1) &= 1 - 3 = -2 \\ L(x) &= 2 - 3(5) = -13 \\ L(x^2) &= \vdots \\ &\vdots \\ L(x^n) &= 2^n - 3(5)^n \end{aligned}$$

What about?

$$L(p(x)) = \frac{d}{dx} p(x)$$

$$\begin{aligned} L(1) &= 0 \\ L(x) &= 1 \\ L(x^2) &= 2x \\ &\vdots \\ L(x^n) &= nx^{n-1} \end{aligned}$$

$$L(p(x)) = \int_0^x p(t)\, dt$$

$$\begin{aligned} L(1) &= x \\ L(x) &= \frac{x^2}{2} \\ &\vdots \\ L(x^n) &= \frac{x^{n+1}}{n+1} \end{aligned}$$

9. $L : \mathbb{R}^2 \to \mathbb{R}^2$ "rotation by $90°$"

$L(a,b) = (-b, a)$

10. If U is finite dimensional

$$\begin{cases} L : U \to \mathbb{R}^{\dim(U)} \\ L(u) = [u]_B \end{cases}$$

B is an ordered basis for U.

Examples Are these maps linear?

1. $L(a,b) = (a+1, 0, b-a)$ $L : \mathbb{R}^2 \to \mathbb{R}^3$

No b/c $L(\underline{0}_U) \neq \underline{0}_V$

$L(0,0) = (1,0,0) \neq (0,0,0)$

2. $L : \mathbb{R}^{2\times 2} \to \mathbb{R}$, $L(A) = \det(A)$.

$L(2A) = \det(2A) = 4\det(A)$.
$\neq 2\det(A)$.

$L\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + L\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 0 + 0$

$L\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$ $\neq$

Kernel and Range

$$L : U \xrightarrow{\text{linear}} V$$

$$\ker(L) = \{ v \in U \mid L(v) = \underline{0} \} \subseteq U$$

$\boxed{Ex}$ $\frac{d}{dx} : \mathbb{P}_3 \longrightarrow \mathbb{P}_3$

$$\ker\left(\frac{d}{dx}\right) = \left\{ p(x) \in \mathbb{P}_3 \;\middle|\; \frac{d}{dx} p(x) = 0 \right\}$$

$$= \{ \text{Constant polynomials} \} = \mathbb{P}_0$$

$$\int_0^x : \mathbb{P}_3 \longrightarrow \mathbb{P}_4$$

$$\ker\left(\int_0^x\right) = \left\{ p(x) \in \mathbb{P}_3 \;\middle|\; \underset{\text{for all } x}{\int_0^x p(x)\,dx = 0} \right\}$$

$$= \{ 0 \}$$

$$\int_0^x (a_0 + a_1 t + a_2 t^2 + a_3 t^3)\,dt = a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3} + a_3 \frac{x^4}{4} = 0$$

$\boxed{Ex}$ Given $A \in \mathbb{R}^{n \times m}$, define

$$\begin{cases} L(x) = A x, & x \in \mathbb{R}^m \\ L : \mathbb{R}^m \longrightarrow \mathbb{R}^n \end{cases}$$

$$L(x+y) = A(x+y) = Ax + Ay = L(x) + L(y)$$

$$\underline{\text{linear}}\ \checkmark \left\} L(\alpha x) = A(\alpha x) = \alpha (Ax) = \alpha L(x)\right.$$

$$\ker(L) = \{x \in \mathbb{R}^m \mid L(x) = \underline{0}\}$$

$$= \{x \in \mathbb{R}^m \mid Ax = \underline{0}\}$$

$$= \text{Null}(A).$$

Proposition $\ker(L)$ is a subspace of U.

$$L : U \xrightarrow{\text{linear}} V$$

Proof. Let $u_1, u_2 \in \ker(L)$. This means

$$L(u_1) = L(u_2) = \underline{0}$$

$$L(u_1 + u_2) = L(u_1) + L(u_2) = \underline{0} + \underline{0} = \underline{0}$$

$$\Rightarrow u_1 + u_2 \in \ker(L).$$

$$L(\alpha u_1) = \alpha L(u_1) = \alpha \cdot \underline{0} = \underline{0}$$

Range (or image)

$$\text{ran}(L) := \{L(u) \mid u \in U\} \subseteq V$$

Prop. $\underline{\text{ran}(L) \text{ is a subspace of } V.}$

Proof

$v_1, v_2 \in \text{ran}(L) \Rightarrow \exists\, u_1, u_2 \in U$ such that $L(u_1) = v_1$, $L(u_2) = v_2$

$$\underbrace{L(u_1) + L(u_2)}_{L(u_1+u_2)} = v_1 + v_2$$

$\Rightarrow v_1 + v_2 \in ran(L)$.

Nullity $(L) := \dim(\ker(L))$

rank $(L) := \dim(ran(L))$

$\boxed{Ex}$ $L_A(x) = \overset{n\times m}{A}x$

$\text{rank}(L_A) = \dim(ran(L)) = \text{rank}(A)$.

Col(A)

$\text{nullity}(L_A) = \dim(\ker(L)) = \dim(\text{Null}(A))$
$= \text{nullity}(A)$

$$\{Ax \mid x \in \mathbb{R}^m\} = Col(A)$$

$$A\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = x_1\, Col_1(A) + x_2\, Col_2(A) + \cdots + x_m\, Col_m(A)$$

$$\Rightarrow \{Ax \mid x \in \mathbb{R}^m\} = \text{span}\{Col_1(A), ---, Col_m(A)\}$$
$$= Col(A).$$

<u>bijection</u> : one-to-one (injective)
+
onto (Surjective)

$L: \underline{U} \xrightarrow{\text{Linear}} \underline{V}$ that is bijective is called an isomorphism.

onto: $\operatorname{ran}(L) = V$

one-to-one: $\boxed{L(u) = L(v) \Rightarrow u = v}$

which means if

$\underbrace{L(u) - L(v)} = 0$ then $u - v = 0$

$L(\underbrace{u-v}_{w}) = 0$ then $\underbrace{u-v}_{w} = 0$

$\underbrace{L(w) = 0}_{w \in \ker(L)} \Longrightarrow w = 0$

So one-to-one means $\ker(L) = \{\underline{0}\}$

Isomorphism: A linear transformation that is bijective.

Example $L: \mathbb{P}_1(x) \to \mathbb{R}^2$

$p(x) = a + bx$ $\quad L(p(x)) = (\underline{p(1), p'(2)})$

Is L linear?

$$L(p(x)+q(x)) = ((p+q)(1),\ (p+q)'(2))$$
$$= (p(1)+q(1),\ p'(2)+q'(2))$$
$$= (p(1), p'(2)) + (q(1), q'(2))$$
$$= L(p) + L(q)$$

Do the same for $L(k\,p(x))$.

(for example
$L(x) = (1,1)$, $L(1) = (1,0)$)

So, yes!

② Is L an isomorphism?

- Is L one to one?

$$\ker(L) = \{p(x) \in \mathbb{P}_1 \mid p(1)=0,\ p'(2)=0\}$$

$p(x) = a + bx$ $\begin{cases} p(1) = a+b = 0 \\ p'(2) = b = 0 \end{cases}$ $\Rightarrow a=b=0$

$$\ker(L) = \{0\}$$

L is injective.

– Is $L: \mathbb{P}_1 \to \mathbb{R}^2$ onto?

Let $(r,s) \in \mathbb{R}^2$. Can we find $p(x) \in \mathbb{P}_1$ such that

$a+bx$

$$L(p(x)) = (r,s)$$

$$L(a+bx) = (a+b,\ b) = (r,s)$$

$$\Rightarrow \quad b = s, \quad a = r-s$$

$$L(r-s+sx) = (r,s).$$

$\Rightarrow$ L is onto.

$\Rightarrow$ L is an isomorphism.

We say U, V are isomorphic $\boxed{U \simeq V}$

if there exists an isomorphism

$$L: U \to V.$$

Theorem $\quad U \simeq V \iff \dim U = \dim V$