Ex $V = \mathbb{R}^3$

$B = \{(2,0,0), (0,1,0), (0,1,1)\}$ is a basis for $\mathbb{R}^3$.

1) B is a spanning set.

2) B is linearly independent.

1) Let $(a,b,c) \in \mathbb{R}^3$. We need to show there exist $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$(a,b,c) = \alpha(2,0,0) + \beta(0,1,0) + \gamma(0,1,1).$$

$$= (2\alpha, \beta+\gamma, \gamma)$$

$$\underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{invertible}} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

invertible: $\det(A) = 2(1)(1) = 2 \neq 0$

$$\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

2) B is linearly independent:

$$(0,0,0) = \alpha(2,0,0) + \beta(0,1,0) + \gamma(0,1,1)$$

the previous formula gives $\underline{\alpha = \beta = \gamma = 0}$.

Example (Standard basis)

1) $V = \mathbb{R}^n$, $E = \{e_1 = (1,0,\dots,0),\ e_2 = (0,1,0,\dots,0), \dots$
$e_n = (0,0,\dots,0,1)\}$

$x = (x_1,\dots,x_n) \in \mathbb{R}^n$ $\qquad x = \sum_{j=1}^{n} x_j e_j$

$c_1 e_1 + \dots + c_n e_n = 0 \implies (c_1,\dots,c_n) = 0 \to c_1 = \dots = c_n = 0$

2) $V = \mathbb{R}^{2\times 2}$ $\qquad E = \Big\{ \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{e_1}, \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{e_2}, \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{e_3}, \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{e_4} \Big\}$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a e_1 + b e_2 + c e_3 + d e_4$

3) $V = \mathbb{P}_n[x] = \{$ polynomials of degree at most $n \}$

$a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n \in V$ $\qquad E = \{1, x, x^2, \dots, x^n\}$

$a_0 (\underline{1}) + a_1 (\underline{x}) + \dots + a_n (\underline{x^n})$

4) $V = \{0\}$, $\quad E = \phi$

Ex $\quad V := \{ A \in \mathbb{R}^{2\times 2} \mid tr(A) = 0 \}$. Find a basis for

$V = \Big\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \Big| \ a, b, c \in \mathbb{R} \Big\}$

$\begin{bmatrix} a & b \\ c & -a \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix} + \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix}$

$$= a\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} + b\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} + c\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}$$

$\Longrightarrow$ $B := \left\{ \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}, \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix} \right\}$ is a spanning set for V. To show linear independence:

let $\underbrace{\alpha\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} + \beta\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} + \gamma\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}}_{\begin{bmatrix}\alpha & \beta\\ \gamma & -\alpha\end{bmatrix}} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$

$$\begin{bmatrix}\alpha & \beta\\ \gamma & -\alpha\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} \Rightarrow \alpha = \beta = \gamma = 0$$

Proposition (optimality) TFAE for $\overset{\text{subset}}{S} \subseteq V$.

1) S is a basis.
2) S is a minimal spanning set
3) S is a maximal linearly independent set.

Example: The set B we proved to be a basis for $\mathbb{R}^3$ in one of the examples above, can be proved to be a basis by only proving it is linearly indep. (or only spanning) b/c $\mathbb{R}^3$ already has a standard basis with three elements e_1, e_2, e_3.

Replacement Theorem

Let S be a spanning set for a vector space V.

Then any set T with more number of elements than S must be dependent

$$|T| > |S| \Rightarrow T \text{ is dependent.}$$

Corollary: If B and C are bases for V, then

$$|B| = |C|.$$

Proof. $|B| \le |C|$ b/c otherwise B is dependent by Replacement theorem since C is spanning.

by symmetry $|B| = |C|$. ∎

Def'n This common size of bases for a vector space V is called dimension of V : $\dim V$.

Ex

- $\dim \{0\} = 0$
- $\dim \mathbb{R}^n = n$

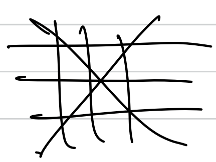
$\dim \mathbb{R}^{m\times n} = mn$

- $\dim(\text{Span}\{u\}) = \begin{cases} 1 & u \neq 0 \\ 0 & u = 0 \end{cases}$
- $\dim(\text{Span}\{u,v\}) = \begin{cases} 2, & u, v \text{ lin indep.} \\ 1, & u, v \text{ lin. dep.} \\ 0, & u = v = 0 \end{cases}$
- $\dim(\mathbb{P}_n[x]) = n+1$
- $\dim(\mathbb{P}[x]) = \infty$

Exercise Magic square

$M = \{3\times 3 \text{ magic squares}\}$:

sum of each row $= 0$
sum of " column $= 0$
sum of diagonals $= 0$



Find a basis for $M \Rightarrow \dim M = ?$

$$\begin{matrix} a & b & c \\ d & e & f \\ g & h & i \end{matrix}$$

Row, Column and Null spaces

"Rank"

Def'n. Let $A \in \mathbb{R}^{m \times n}$. The Column space of A

$$\underline{Col(A)} := span\{\text{Columns of } A\} \overset{\text{Subspace}}{\subset} \mathbb{R}^m$$

Row space

$$\underline{Row(A)} := span\{\text{rows of } A\} \overset{\text{subspace}}{\subset} \mathbb{R}^n$$

null space

$$Null(A) := \{x \in \mathbb{R}^n \mid Ax = 0 \in \mathbb{R}^m\} \overset{\text{Subspace}}{\subset} \mathbb{R}^n$$

to find: $\quad [A \mid \underset{\text{column}}{0}]$ augmented

then Find the solution set

Remark. $Col(A^T) = Row(A)$

Def. Rank of a matrix

$$rank(A) := dim(Col(A))$$

Def. Nullity of a matrix

$$nullity(A) := dim(Null(A))$$

Proposition Let R be an echelon form of $\underline{A}$. Then

1) The columns of A corresponding to the leading columns of R form a basis for $Col(A)$.

2) The non-zero rows of R form a basis for $Row(A)$.

$\boxed{Ex}$ $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \in \mathbb{R}^{2\times 3}$

$\longrightarrow R = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$: RREF

$\Rightarrow$ A basis for $Row(A)$: $\{(1,0,-1), (0,1,2)\}$

$\Rightarrow$ A basis for $Col(A)$: $\{ \underline{(1,4)}, \underline{(2,5)} \}$

$\Rightarrow rank(A) = dim(Col(A)) = 2$

Corollary: $dim(Row(A)) = dim(Col(A)) = rank(A)$

* If we need to find a basis for $Row(A)$ that has to be from the same rows of A just find an echelon form of A^T, then

study $Col(A^T)$, and then transpose!

Coordinates

Let V be a vector space and fix an ordered basis:

$$\underline{\underline{B = (b_1, b_2, \dots, b_n)}} \quad \left(= [b_1, b_2, \dots, b_n] \right)$$

Then given $v \in V$, there is a unique representation of v as a linear combination of vectors in B.

$$v = \underline{v_1 b_1 + \dots + v_n b_n} = \underline{w_1 b_1 + \dots + w_n b_n}$$

$$(v_1 - w_1) b_1 + \dots + (v_n - w_n) b_n = 0$$

$$\Rightarrow \quad v_j - w_j = 0 \qquad j = 1, \dots, 2$$

exactly b/c $b_1, \dots, b_n$ linearly independent.

Notation $\underbrace{[v]_B = (v_1, \ldots, v_n)}_{\text{the coordinate of } v \text{ w.r.t. } B} \in \mathbb{R}^n$

map : $V \longrightarrow \mathbb{R}^n$

$v \longmapsto \underbrace{(v_1, \ldots, v_n)}_{\text{depends on choice of orderd basis for } V.}$