Lecture 5

Ex Expand the determinant along the second row.

$$\det\begin{bmatrix} 1^{+} & 2 & 3 \\ 2^{-} & 0^{+} & -2^{-} \\ 2 & 1 & 4 \end{bmatrix} = -\overset{a_{21}}{2}\,\det\overbrace{\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}}^{A_{21}} + \overset{a_{22}}{0}\,\det\overbrace{\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}}^{A_{22}}$$

$$-\overset{a_{23}}{(-2)}\,\det\underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}}_{A_{23}}$$

$$= -2(8-3) + 2(1-4) = -10 - 6 = -16.$$

Note: Picking a row that contains zero's is better b/c it reduces the number of finding the cofactor determinants.

Fact The determinant can be found using cofactor expansion along any column

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j}\,\overset{A(i,j)}{a_{ij}}\,\det|A_{ij}|$$

Ex $\begin{vmatrix} 3 & 2 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{vmatrix} = +3 \begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} = 3(2)(1) = 6$

all 0's

Fact The determinant of an upper triangular matrix is the product of the diagonal elements.

upper triangular: $A(i,j) = 0$ for all $j < i$

(or lower)

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ -1 & \pi & 0 & 0 \\ -2 & 3 & e & 0 \\ 0 & 1 & 2 & x \end{vmatrix} = 1 \cdot \pi \cdot e \cdot x$$

lower triangular

Row operations and the determinant.

Proposition $A \xrightarrow{R_i \leftrightarrow R_j} B$, then $\det(A) = -\det(B)$

$\begin{vmatrix} c & d \\ a & b \end{vmatrix} = bc - ad$ $\qquad$ $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$$A = \begin{array}{c} \\ \begin{pmatrix} \overset{+}{a} & b & c \\ d^{-} & e^{+} & f^{-} \\ g & h & i \end{pmatrix} \end{array} \xrightarrow{R_3 \leftrightarrow R_1} \begin{pmatrix} g & h & i \\ d & e & f \\ a & b & c \end{pmatrix} = B$$

2^{nd} row

$$\det(B) = -d \begin{vmatrix} h & i \\ b & c \end{vmatrix} + e \begin{vmatrix} g & i \\ a & c \end{vmatrix} - f \begin{vmatrix} g & h \\ a & b \end{vmatrix}$$

$$= -d \left(- \begin{vmatrix} b & c \\ h & i \end{vmatrix} \right) + e \left(- \begin{vmatrix} a & c \\ g & i \end{vmatrix} \right)$$

$$- f \left(- \begin{vmatrix} a & b \\ g & h \end{vmatrix} \right)$$

$$= - \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

Prop If A is a square matrix such that the i^{th} row is the same as j^{th} row, then

$\det(A) = 0$

$$A \xrightarrow{R_i \leftrightarrow R_j} A \quad \Rightarrow \det(A) = -\det(A)$$

$$\Rightarrow \det(A) = 0$$

Prop. If $A \xrightarrow{R_i \leftarrow cR_i} B$

then $\det(B) = c\det(A)$

Proof.

$$\det(B) = \sum_{j=1}^{n} (-1)^{i+j} B(i,j) \det(B_{ij})$$

the ith row expansion

$$= \sum_{j=1}^{n} (-1)^{i+j} \left[c A(i,j) \right] \det(A_{ij})$$

$$= c \sum_{j=1}^{n} (-1)^{i+j} A(i,j) \det(A_{ij})$$

$$= c \det(A).$$

Prop The determinant is "additive with respect to rows"

R_1: the first row, etc.

$$\underbrace{\det \begin{bmatrix} R_1 \\ \vdots \\ R_i + \tilde{R}_i \\ \vdots \\ R_n \end{bmatrix}}_{C} = \underbrace{\det \begin{bmatrix} R_1 \\ \vdots \\ R_i \\ \vdots \\ R_n \end{bmatrix}}_{A} + \underbrace{\det \begin{bmatrix} R_1 \\ \vdots \\ \tilde{R}_i \\ \vdots \\ R_n \end{bmatrix}}_{B}$$

$$\begin{array}{c} R_1 \\ R_2 + \tilde{R}_2 \\ R_3 \end{array} \begin{vmatrix} a & b & c \\ d_1+d_2 & e_1+e_2 & f_1+f_2 \\ g & h & i \end{vmatrix} = \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \begin{vmatrix} a & b & c \\ d_1 & e_1 & f_1 \\ g & h & i \end{vmatrix} + \begin{vmatrix} a & b & c \\ d_2 & e_2 & f_2 \\ g & h & i \end{vmatrix} \begin{array}{c} \\ \tilde{R}_2 \\ \\ \end{array}$$

Proof

$$\det(C) \underset{\text{ith row}}{=} \sum_{j=1}^{n} (-1)^{i+j} C(i,j) \det(C_{ij})$$

$$= \sum_{j=1}^{n} (-1)^{i+j} \left[A(i,j) + B(i,j) \right] \underbrace{\det(C_{ij})}_{= \det(A_{ij}) = \det(B_{ij})}$$

$$= \sum_{j=1}^{n} (-1)^{i+j} A(i,j) \det(A_{ij}) + \sum_{j=1}^{n} (-1)^{i+j} B(i,j) \det(B_{ij})$$

$$= \det(A) + \det(B)$$

$$\begin{vmatrix} a^{+} & b & c \\ d+\tilde{d}^{-} & e+\tilde{e}^{+} & f+\tilde{f}^{-} \\ g & h & i \end{vmatrix} = -(d+\tilde{d}) \begin{vmatrix} b & c \\ h & i \end{vmatrix} + (e+\tilde{e}) \begin{vmatrix} a & c \\ g & i \end{vmatrix} - (f+\tilde{f}) \begin{vmatrix} a & b \\ g & h \end{vmatrix}$$

$$= \left(-d \begin{vmatrix} b & c \\ h & i \end{vmatrix} + e \begin{vmatrix} a & c \\ g & i \end{vmatrix} - f \begin{vmatrix} a & b \\ g & h \end{vmatrix} \right)$$

$$+ \left(-\tilde{d} \begin{vmatrix} b & c \\ h & i \end{vmatrix} + \tilde{e} \begin{vmatrix} a & c \\ g & i \end{vmatrix} - \tilde{f} \begin{vmatrix} a & b \\ g & h \end{vmatrix} \right)$$

$$= \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} a & b & c \\ \tilde{d} & \tilde{e} & \tilde{f} \\ g & h & i \end{vmatrix}$$

Ex

$$\begin{vmatrix}1&2&3\\4+1&5+2&6+3\\7&8&9\end{vmatrix} = \begin{vmatrix}1&2&3\\4&5&6\\7&8&9\end{vmatrix} + \underbrace{\begin{vmatrix}1&2&3\\1&2&3\\7&8&9\end{vmatrix}}_{=\,0}$$

$R_2 \leftarrow R_2 - R_1$

$R_2 \leftarrow R_2 + R_1$

Prop. If $A \xrightarrow{R_i \leftarrow R_i + cR_k} B$

then $\det(A) = \det(B)$.

Proof

$$\det(B) = \begin{vmatrix}R_1\\ \vdots \\ R_k \\ R_i + cR_k \\ \vdots \\ R_n\end{vmatrix} = \underbrace{\begin{vmatrix}R_1\\ \vdots \\ R_k \\ \vdots \\ R_i \\ \vdots \\ R_n\end{vmatrix}}_{} + \underbrace{\begin{vmatrix}R_1\\ \vdots \\ R_k \\ cR_k \\ \vdots \\ R_n\end{vmatrix}}_{}$$

$$= \det(A) + c\underbrace{\begin{vmatrix}R_1\\ \vdots \\ R_k \\ \vdots \\ R_k \\ \vdots \\ R_n\end{vmatrix}}_{=\,0} = \det(A).$$

flattening notation

Summary

$A \xrightarrow{R_i \leftrightarrow R_j} B \Rightarrow \det(B) = -\det(A)$

$A \xrightarrow{R_i \leftarrow cR_i} B \overset{c \neq 0}{\Rightarrow} \det(B) = c \cdot \det(A)$

$A \xrightarrow{R_i \leftarrow R_i + cR_j} B \Rightarrow \det(B) = 1 \cdot \det(A)$

Remark 1: Notice $\det(B) = \alpha \det(A)$, $\alpha = -1, c, \text{ or } 1$ $\alpha \neq 0$

Remark 2: Any square matrix in row echelon form is upper triangular.

Ex Find the determinant using elimination

$$\begin{vmatrix} 2 & 1 & 6 \\ -1 & 2 & 0 \\ 2 & 0 & 6 \end{vmatrix} \overset{R_1 \leftrightarrow R_3}{=} - \begin{vmatrix} 2 & 0 & 6 \\ -1 & 2 & 0 \\ 2 & 1 & 6 \end{vmatrix}$$

$$\xrightarrow{R_1 \leftarrow \frac{1}{2} R_1} = -2 \begin{vmatrix} 1 & 0 & 3 \\ -1 & 2 & 0 \\ 2 & 1 & 6 \end{vmatrix} \underset{R_3 \leftarrow R_3 - 2R_1}{\overset{R_2 \leftarrow R_2 + R_1}{=}} -2 \begin{vmatrix} 1 & 0 & 3 \\ 0 & 2 & 3 \\ 0 & 1 & 0 \end{vmatrix}$$

$$\overset{R_2 \leftrightarrow R_3}{=} -(-2) \begin{vmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 2 & 3 \end{vmatrix} \overset{R_3 \leftarrow R_3 - 2R_2}{=} 2 \underbrace{\begin{vmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{vmatrix}}_{\text{upper triangular}}$$

$$= 2(1)(1)(3) = 6.$$

Corollary If E is an elementary matrix then $\det(EA) = \det(E)\det(A)$.

Proof. $[I \mid A] \xrightarrow{e} [E \mid EA]$

(B as above
$A \xrightarrow{e} EA$
$\det(EA) = \alpha \det(A)$)

there exists a nonzero α,

$$\det(E) = \alpha \underbrace{\det(I)}_{1} = \alpha$$

$$\Rightarrow \det(EA) = \alpha \det(A) = \det(E)\det(A). \square$$

$$\begin{cases} \alpha = -1 & \text{if } e = R_i \leftrightarrow R_j \\ \alpha = c \neq 0 & \text{if } e = R_i \leftarrow cR_i \\ \alpha = 1 & \text{if } e = R_i \leftarrow R_i + cR_j \end{cases}$$

Prop. A square matrix is invertible iff its determinant is nonzero.

Proof. Apply Gauss-Jordan reduction, get

$$R := RRef(A).$$

$$\det(A) = \underbrace{\text{nonzero constant}}_{\alpha} \cdot \det(A).$$

$$\begin{array}{ccccccc} A & \xrightarrow{e_1} & E_1A & \xrightarrow{e_2} & E_2E_1A \to \cdots & \xrightarrow{e_n} & E_nE_{n-1}\cdots E_1A = R = RRef(A) \\ \downarrow \det & & \downarrow & & \downarrow & & \downarrow \\ \det(A) & & \alpha_1 \det A & & \alpha_2\alpha_1 \det(A) & & \underbrace{\alpha_n \alpha_{n-1} \cdots \alpha_1}_{\alpha} \det(A) \end{array}$$

Here $\alpha_j = \det(E_j) \neq 0$

$$\det(R) = \alpha \cdot \det(A).$$

- If A is invertible, $R = I \Rightarrow \det(R) = 1 = \alpha \det(A)$

$$\Rightarrow \det(A) = \frac{1}{\alpha} \neq 0.$$

- If A is singular, the R has at least one zero row $\Rightarrow \det(R) = 0 = \underset{\neq 0}{\alpha} \cdot \det(A)$.

$$\Rightarrow \det(A) = 0$$

Ex

$A = \begin{pmatrix} 1 & 2 \\ 3 & x \end{pmatrix}$ is singular precisely when $\det(A) = 0$

$1(x) - 2(3) = 0$

$\Rightarrow \boxed{x = 6}$

Proposition $\det(AB) = \det(A)\det(B)$

Proof. If A is not invertible, then $\det(A) = 0$, and also AB is not invertible [If AB has an inverse C, then

$$(AB)C = I \Rightarrow A(\underbrace{BC}_{\text{inverse of } A}) = I$$
]

$$\underbrace{\det(AB)}_{0} = \underbrace{\det(A)}_{0} \cdot \det(B) \qquad 0 = 0 \checkmark$$

,

- Suppose A is invertible:

$$A = E_1^{-1} E_2^{-1} \cdots E_n^{-1} \Rightarrow \det(A) = \det(E_1^{-1}) \cdots \det(E_n^{-1})$$

$$\det(AB) = \det(E_1^{-1} \cdots E_n^{-1} B) = \underbrace{\det(E_1^{-1}) \cdots \det(E_n^{-1})}\det(B)$$

$$= \det(A)\det(B)$$

Corollary: If A is invertible

$$\boxed{\det(A^{-1}) = \frac{1}{\det(A)}}$$