<u>Ex</u> $L : \mathbb{R}^3 \to \mathbb{R}^3$: 90° rotation around the z-axis

followed by the orthogonal projection to the yz-plane.

L_1 : 90° rotation around z-axis , L_2 : orthogonal projection to the yz-plane

$L_1, L_2 : \mathbb{R}^3 \to \mathbb{R}^3$ linear.

$$L = L_2 \circ L_1 = L_2 L_1 \quad \text{linear}$$

$$[L] = [L]_E = \left([L(e_1)]_E \ [L(e_2)]_E \ [L(e_3)]_E \right)_{3\times 3}$$

$L(e_1) = L(1,0,0)$

$$(1,0,0) \xrightarrow{L_1} (0,1,0) \xrightarrow{L_2} (0,1,0)$$

$$L$$

$$(0,1,0) \xrightarrow{L_1} (-1,0,0) \xrightarrow{L_2} (0,0,0)$$

$$(0,0,1) \xrightarrow{L_1} (0,0,1) \xrightarrow{L_2} (0,0,1)$$

$$A = [L] = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$L(a,b,c) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ a \\ c \end{bmatrix} = (0, a, c)$$

$$\operatorname{ran}(L) = \operatorname{Col}(A) = \operatorname{Span}\{(0,1,0), (0,0,0), (0,0,1)\}$$

$$= \operatorname{Span}\{\underbrace{(0,1,0), (0,0,1)}_{\text{lin. indep.}}\}$$

$$= yz\text{-plane}$$

$$\ker(L) = \text{Null}(A) = \{(a,b,c) \in \mathbb{R}^3 \mid A(a,b,c) = \underline{0}\}$$

$$= \{(a,b,c) \mid (0,a,c) = (0,0,0)\}$$

$$= \{(0,b,0) \mid b \in \mathbb{R}\} = \text{span}\{(0,1,0)\}$$

$$= y\text{-axis}$$

$$\text{rank}(L) = \dim(yz\text{-plane}) = 2$$

$$\text{nullity}(L) = \dim(y\text{-axis}) = 1$$

$$2+1 = 3 = \dim(\text{dom}(L)) = \dim(\mathbb{R}^3)$$

$\boxed{\text{dimension theorem}}$ ✓

$\boxed{\text{Ex}}$ $L : \mathbb{P}_2 \to \mathbb{R}^2$ $\quad L(p(x)) = (p(2) + p'(3), p''(1))$

$$L(p+q) = (p(2) + q(2) + p'(3) + q'(3),\ p''(1) + q''(1))$$

$$= L(p) + L(q)$$

$$L(\alpha p) = \alpha L(p)$$

L is linear.

$$L(1) = (1+0, 0) = (1,0)$$

$$L(x) = (2+1, 0) = (3,0)$$

$$L(x^2) = (4+6, 2) = (10,2)$$

$E = (1, x, x^2)$: standard basis for $\mathbb{P}_2$

$F = (e_1, e_2)$: " " " $\mathbb{R}^2$

$$[L]_E^F = \Big[[L(1)]_F \quad [L(x)]_F \quad [L(x^2)]_F \Big] \in \mathbb{R}^{2\times 3}$$

leading

$$= \begin{bmatrix} 1 & 3 & 10 \\ 0 & 0 & 2 \end{bmatrix}$$

free

already echelon form

$$\text{Col}([L]_E^F) = \text{span}\{(1,0), (10,2)\} \Rightarrow \boxed{\text{rank}(L) = 2}$$

$$\text{ran}(L) = \text{Span}\{(1,0),\ (10,2)\}$$

$$\text{Null}\left([L]^F_E\right) = \left\{ (x_1,x_2,x_3) \;\middle|\; [L]^F_E \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

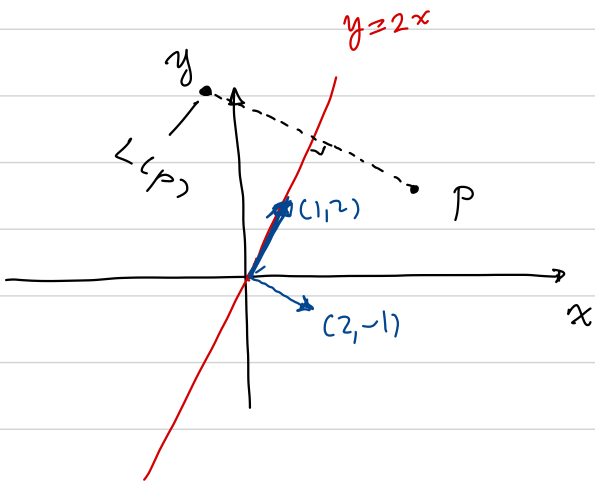
$$= \{ (x_1,x_2,x_3) \mid x_1 + 3\underset{\downarrow\ \text{free}}{x_2} + 10x_3 = 0 \ \ \& \ \ 2x_3 = 0 \}$$

$$= \{ (-3x_2, x_2, 0) \mid x_2 \in \mathbb{R} \} = \text{Span}\{ (-3,1,0) \}$$

$$\Rightarrow \text{Null}(L) = \text{Span}\{-3+x\} \Rightarrow \boxed{\text{nullity}(L) = 1}$$

$$\therefore \text{rank}(L) + \text{nullity}(L) = 2+1 = 3 = \dim \mathbb{P}_2 .$$

$\boxed{\text{Ex}}$ $L : \mathbb{R}^2 \to \mathbb{R}^2$: reflection of the plane with respect to the line $y = 2x$



$$L(a,b) = [L]^E_E \begin{bmatrix} a \\ b \end{bmatrix}$$

standard basis

Select a basis $B = \left(\underset{b_1}{(1,2)}, \underset{b_1}{(2,-1)} \right)$

$[L]^B_B$ is easy to write:

$L(1,2) = (1,2)$, $L(2,-1) = -(2,-1) = (-2,1)$

$$[L]^B_B = [L]_B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$[L]_E^E = T_B^E \, [L]_B^B \, T_E^B \rightarrow \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}^{-1} = \frac{1}{-5}\begin{bmatrix} -1 & -2 \\ -2 & 1 \end{bmatrix}$$

$$[[b_1]_E \; [b_2]_E] \qquad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}$$

$$[L]_E^E = \frac{1}{5}\begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}$$

$$L(a,b) = \frac{1}{5}\begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -3a+4b \\ 4a+3b \end{bmatrix}$$

$$= \left(\frac{-3a+4b}{5}, \frac{4a+3b}{5} \right)$$

Recall

transition matrix $\qquad U$ with two bases $\mathcal{B}, E$

$(e_1, \ldots, e_m)$

$v \in U$

$$[v]_{\mathcal{B}} = T_E^{\mathcal{B}} [v]_E$$

$\mathcal{B} = (b_1, \ldots, b_m)$

$$[\,[e_1]_{\mathcal{B}} \cdots [e_m]_{\mathcal{B}}\,]$$

$$v = \sum_{j=1}^{m} \alpha_j e_j \qquad [v]_E = (\alpha_1, \ldots, \alpha_m)$$

$$[v]_{\mathcal{B}} = \sum_{j=1}^{m} \alpha_j [e_j]_{\mathcal{B}} = T_E^{\mathcal{B}} [v]_E$$

$$[v]_{\mathcal{B}} = \left(T_B^E\right)^{-1} [v]_E$$

A Transition matrix is a <u>particular matrix representation</u>

$I : U \longrightarrow U$ $\qquad I(v) = v$

I: Identity; U with basis E; U with basis B

$$\underbrace{[I]^E_B}_{T^E_B} [v]_B = [I(v)]_E = [v]_E$$

$$\underbrace{[I]^B_E}_{T^B_E} [v]_E = [I(v)]_B = [v]_B$$

$$L : U \to V$$

$$\begin{tikzcd}
(U,B) \arrow[r, "{[L]^C_B}"] & (V,C) \\
(U,E) \arrow[u, "{T^B_E = [I_U]^B_E}"] \arrow[r, "{[L]^F_E}"'] \arrow[ur, phantom, "(C)"] & (V,F) \arrow[u, "{[I_V]^C_F = T^C_F}"']
\end{tikzcd}$$

$$I_V L = L I_U$$

let $u \in U$: $\qquad L I_U(u) = L(u)$

$\qquad\qquad\qquad I_V L(u) = L(u)$

$$[L]^C_B T^B_E = T^C_F \underbrace{[L]^F_E}_{\text{easy}}$$

inverse is easy

if E, F are standard then some are easier to find.

$$[L]^C_B$$

$$[L]_B^C \, (T_B^E)^{-1} = (T_C^F)^{-1} \, [L]_E^F$$

$$[L]_B^C = (T_C^F)^{-1} \, [L]_E^F \, T_B^E$$

"difficult" easy

$$[L]_E^F = T_C^F \, [L]_B^C \, (T_B^E)^{-1}$$

Similarity

$L : U \longrightarrow U$ linear

B, C : two ordered bases for U

$$[L]_B^B = (T_B^C)^{-1} \, [L]_C^C \, T_B^C$$

This motivates:

Two matrices $A, B \in \mathbb{R}^{n \times n}$ are called similar

if $A = P^{-1} B P$ for some invertible matrix P. $A \sim B$

$\boxed{\text{Ex}}$ $[L]_B^B \sim [L]_C^C$ (with $P = T_B^C$).

Similarity is an equivalence relation.

1) reflexive $A \sim A$ ✓ $(P = I)$

2) symmetric : $A \sim B \Rightarrow B \sim A$

$$A = P^{-1} B P \Rightarrow P A P^{-1} = B$$
$$\Rightarrow (P^{-1})^{-1} A (P^{-1}) = B$$ ✓

3) transitive $A \sim B$, $B \sim C \Rightarrow A \sim C$

$A = P^{-1} B P$ $\qquad$ $B = Q^{-1} C Q$

$$A = P^{-1} (Q^{-1} C Q) P$$
$$= (P^{-1} Q^{-1}) C (Q P)$$
$$= (QP)^{-1} C (QP)$$

Eigenvector

$$\begin{cases} L : V \longrightarrow V & \text{linear operator} \\ B = (b_1, \dots, b_m) & \text{ordered basis} \end{cases}$$

$$M := [L]_B^B = [L]_B$$

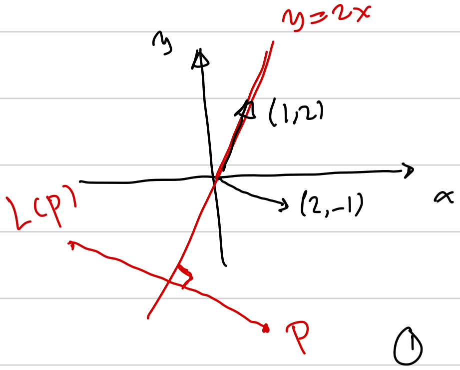
Recall $M = \left[[L(b_1)]_B \; [L(b_2)]_B \; \cdots \; [L(b_m)]_B \right]$

Def'n If $\boxed{L(v) = \lambda v}$ for some $\underline{\underline{v \neq 0}}$ in V and some $\lambda \in \mathbb{R}$,

then λ is called an <u>eigenvalue</u> of L

v is a corresponding <u>eigenvector</u>.

Ex "Reflection about $y = 2x$"



$L: \mathbb{R}^2 \to \mathbb{R}^2$

$L(a,b) = \lambda (a,b)$

① $L(\underbrace{2,-1}_{v_1 \neq 0}) = \underbrace{(-)}_{\lambda_1 = -1}(2,-1)$

$\begin{cases} L(v_1) = -v_1 \\ \lambda_1 = -1 \end{cases}$

② $L(\underbrace{1,2}_{v_2 \neq 0}) = (1,2)$

$\begin{cases} L(v_2) = v_2 \\ \lambda_2 = 1 \end{cases}$

$\boxed{\text{Ex}}$ $V = \{$ ~~twice~~ infinitely many times differentiable functions $f: \mathbb{R} \to \mathbb{R} \} = C^\infty(\mathbb{R})$

Vector space

$\underline{L(f) = f'}$ linear operator:

$L(f) = f' \stackrel{?}{=} \lambda f$

$L(e^{10x}) = \underset{\lambda}{\underset{\downarrow}{10}}\, e^{10x}$ $\qquad$ $L(e^{\lambda x}) = \lambda e^{\lambda x}$

$\left\{\begin{array}{l}\text{any real number is an eigenvalue of} \\ L(f)=f', \quad (f=e^{\lambda x} \text{ eigenvector}).\end{array}\right.$

$y' = \lambda y$, $\quad \frac{y'}{y} = \lambda$, $\quad \underset{=\lambda}{(\ln y)'} \overset{\#}{\underset{0}{}}$ $\quad \ln y = \lambda x$, $\quad y = Ce^{\lambda x}$

$\boxed{Ex}$ $\qquad \underline{\underline{L(f) = f''}}$

$L(\cos x) = -\cos x$ $\qquad \lambda = -1$, $v = \cos x$ or $\sin x$

$L(\sin x) = -\sin x$

$y'' = \lambda y.$ $\qquad \lambda < 0 \quad \left\{\begin{array}{l}\sin\sqrt{-\lambda}\, x \\ \cos\sqrt{-\lambda}\, x\end{array}\right.$

$\lambda > 0 \qquad \underline{\underline{e}}^{\sqrt{\lambda}\, x}, \; \underline{\underline{e}}^{-\sqrt{\lambda}\, x}$