Lecture 6

Cramer's Rule

Notation $B \in \mathbb{R}^{n \times n}$, $v \in \mathbb{R}^{n \times 1}$: column matrix

Define

$B_i[v]$: Replace the ith column of B by v

Ex

$B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\quad v = \begin{bmatrix} 8 \\ 9 \end{bmatrix}$

$B_1[v] = \begin{bmatrix} 8 & 2 \\ 9 & 4 \end{bmatrix}$ $\quad B_2[v] = \begin{bmatrix} 1 & 8 \\ 3 & 9 \end{bmatrix}$

Ex If $I \in \mathbb{R}^{3 \times 3}$ is the identity, $v = (7, 8, 9)$

$$I_2[v] = \begin{bmatrix} 1 & 7 & 0 \\ 0 & 8 & 0 \\ 0 & 9 & 1 \end{bmatrix} = [e_1 \;\; v \;\; e_3]$$

$$I_1[v] = [v \;\; e_2 \;\; e_3]$$

Lemma $\det(I_i[v]) = v_i = v(i)$

Proof

Cofactor expansion along the ith row gives

$$x(i)\det(I) = x(i).$$

system $\quad Ax = b \qquad A \in \mathbb{R}^{n \times n}$

$\qquad\qquad\qquad\qquad\qquad x, b \in \mathbb{R}^{n \times 1}$

$$\begin{array}{c} \text{i-th column} \\ \downarrow \end{array}$$

$$A\left(I_i(x)\right) = A\,[e_1 \cdots x \cdots e_n]$$

$$= [\underbrace{Ae_1}_{\text{1st column of } A} \cdots \overset{b}{Ax} \cdots \underbrace{Ae_n}_{\text{nth column of } A}]$$

$$A_i[b] =$$

$$A = \underbrace{AI} = A\,[e_1 \cdots e_i \cdots e_n] = [Ae_1 \cdots Ae_i \cdots Ae_n]$$

If $\quad Ax = b \quad$ then

Summary :

$$A\left(I_i(x)\right) = A_i[b]$$

$$\det\left(A\, I_i(x)\right) = \det(A)\,\overbrace{\det\left(I_i(x)\right)}^{x(i)} = \det\left(A_i[b]\right)$$

$$\Rightarrow \quad x(i) = x_i = \frac{\det(A_i[b])}{\det(A)}$$

Cramer's formula

Ex $\begin{cases} x + 2y = \pi \\ 3x + 4y = e \end{cases}$ $\qquad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \pi \\ e \end{bmatrix}$

$$x = \frac{\begin{vmatrix} \pi & 2 \\ e & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} 1 & \pi \\ 3 & e \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}}$$

$$= \frac{4\pi - 2e}{-2} \qquad y = \frac{e - 3\pi}{-2}$$

Classical Adjoint

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$$

$A \in \mathbb{R}^{n \times n}$, classical adjoint of A is defined by

$$\operatorname{adj}(A)(i,j) := (-1)^{i+j} \det(A_{ji})$$

In other words, $\operatorname{adj}(A)$ is the matrix obtained from the transpose of the cofactors.

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \longrightarrow \operatorname{adj}(A) = \begin{bmatrix} \begin{vmatrix} e & f \\ h & i \end{vmatrix} & -\begin{vmatrix} b & c \\ h & i \end{vmatrix} & \begin{vmatrix} b & c \\ e & f \end{vmatrix} \\ -\begin{vmatrix} d & f \\ g & i \end{vmatrix} & \begin{vmatrix} a & c \\ g & i \end{vmatrix} & -\begin{vmatrix} a & c \\ d & f \end{vmatrix} \\ \begin{vmatrix} d & e \\ g & h \end{vmatrix} & -\begin{vmatrix} a & b \\ g & h \end{vmatrix} & \begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{bmatrix}$$

3×3

Observation

$$A \cdot \operatorname{adj}(A) = \operatorname{adj}(A) \cdot A = \begin{bmatrix} \det(A) & 0 & 0 \\ 0 & \det(A) & 0 \\ 0 & 0 & \det(A) \end{bmatrix}$$

$$-a \begin{vmatrix} b & c \\ h & i \end{vmatrix} + b \begin{vmatrix} a & c \\ g & i \end{vmatrix} - c \begin{vmatrix} a & b \\ g & h \end{vmatrix}$$

$$= \det \begin{bmatrix} a & b & c \\ a & b & c \\ g & h & i \end{bmatrix} = 0$$

$$\boxed{A \cdot \operatorname{adj}(A) = \operatorname{adj}(A) \cdot A = \det A \cdot I}$$

for all $A \in \mathbb{R}^{n \times n}$

In Particular, if A is invertible ($\det A \neq 0$)

then

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A).$$

③ Vector spaces

Definition A vector space is set V with

two operations:

- vector addition $u+v \in V$, for all u, v
- Scalar multiplication: $cu \in V$, $c \in \mathbb{R}$, $u \in V$

Such that:

for all

① $\forall\, u, v \in V$: $u+v = v+u$

② $\forall\, u, v, w \in V$: $(u+v)+w = u+(v+w)$

③ $\exists$ a zero vector $\underline{0} \in V$ such that $\underline{0} + u = u$, $\forall u \in V$

"there exists"

opposite

④ $\forall u \in V$ $\exists\, -u \in V$ such that $u + (-u) = \underline{0}$

⑤ $c(u+v) = cu + cv$ $\forall c \in \mathbb{R}$, $\forall u, v \in V$

⑥ $(c+d)v = cv + dv$ $\forall c, d \in \mathbb{R}$, $\forall v \in V$

⑦ $(cd)v = c(dv)$ $\forall c, d \in \mathbb{R}$, $\forall v \in V$

⑧ $1v = v$ $\forall v \in V$

$(1 \in \mathbb{R})$

Every $v \in V$ is called a vector.

Example

- $\mathbb{R}^n$: n-tuples
- $\mathbb{R}^{m \times n}$
- upper triangular $n \times n$ matrices
- $n \times n$ matrices with trace $\underline{\underline{0}}$.
- sequences $\quad \underline{0}$: $0, 0, 0, 0, \ldots$
- Convergent sequences
- Real valued functions with domain $\mathbb{R}$

$$\{ f : \mathbb{R} \to \mathbb{R} \}$$

- Polynomials of degree less than or equal to $\underline{\underline{n}}$.
- $n \times n$ magic squares.
- $n \times n$ magic squares with magic sum $\underline{0}$.

Non-examples (with the usual operations)

- Invertible matrices $(n \times n)$ $\quad \underline{0} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & & 0 \end{bmatrix}_{n \times n}$ is not invertible
- trace $= 1$ $\quad (n \times n)$ $\quad \underline{0}$ has trace $= 0 \neq 1$

· Sequences converging to π.

$\underline{0} = (0, 0, 0, \ldots) \to 0 \neq \pi$

$\underline{0} \notin$ this set.

Define $u - v := u + (-v)$

Proposition $0v = \underline{0} \quad \forall v \in V$.

Proof.

$$0v = 0v + \underline{0} = 0v + (0v - 0v)$$

$$= (0v + 0v) - 0v$$

$$= (0+0)v - 0v$$

$$= 0v - 0v = \underline{0}$$

opposite vector

Prop $-v = (-1)v \quad \forall v \in V$

Proof.

$$(-1)v = \underline{0} + (-1)v$$

$$= (-v + v) + (-1)v$$

$$= -v + [v + (-1)v]$$

$$= -v + [1v + (-1)v]$$

$$= -v + [(1 + (-1))v]$$

$$= -v + 0v = -v + \underline{0} = -v.$$

Example ① $V = \mathbb{R}$ define

$$\begin{cases} u \oplus v = u + v - 1 & u, v \in \mathbb{R} \\ \alpha \odot u = \alpha u - \alpha + 1 & \alpha \in \mathbb{R},\ u \in \mathbb{R} \end{cases}$$

Prove that $(V, \oplus, \odot)$ is a vector space

just a few checks:

$$u \oplus v \overset{?}{=} v \oplus u$$

$$u + v - 1 = v + u - 1$$

$$u \oplus (v \oplus w) = u \oplus (v + w - 1) = u + (v + w - 1) - 1$$

$$= u + v + w - 2$$

$$(u \oplus v) \oplus w = (u + v - 1) \oplus w = u + v - 1 + w - 1$$

$$= u + v + w - 2$$

etc.

what is $\underline{0}$? what is $\ominus u$?

$\underline{0} = x$

$$u \oplus x = u + x - 1 = u$$

$$\Rightarrow \boxed{x = 1}$$

$$\underline{0} = 1$$

$\ominus 7 = y \Rightarrow 7 \oplus y = 7 + y - 1 = \overset{\underline{0}}{1}$

$\Rightarrow y = 2 - 7 = -5$

$\ominus u = 2 - u$.

Example 2 $\quad V = \underline{\underline{(-1, 1)}}$

$$u \oplus v := \frac{u+v}{1+uv}, \quad \alpha \odot u = \frac{(1+u)^\alpha - (1-u)^\alpha}{(1+u)^\alpha + (1-u)^\alpha}$$

$$u \oplus \overset{\underline{0}}{\overset{\parallel}{x}} = u, \quad \forall u$$

$$\frac{u+x}{1+ux} = u \Rightarrow \cancel{u} + x = \cancel{u} + u^2 x$$

$$x = u^2 x$$

$$x(\underbrace{1-u^2}_{>0 \text{ b/c } -1<u<1}) = 0 \Rightarrow x = 0$$

$$\therefore \underline{0} = 0$$

$$y = \ominus u \qquad \frac{u+y}{1+uy} = \underline{0} = 0$$

$$\Rightarrow y = \ominus u = -u$$

$$0.1 \oplus 0.1 = \frac{0.2}{1+0.01} = \frac{0.2}{1.01}$$

$$5 \odot (0.1) = \frac{(1.1)^5 - (0.9)^5}{(1.1)^5 + (0.9)^5}$$

Subspaces

Definition A subspace of a vector space V is a subset W of V that is a vector space as well with the operations inherited from V.

Example: $V = \mathbb{R}^{3\times 3}$

$W_1 = \{3\times 3 \text{ upper triangle matrices}\}$

$W_2 = \{3\times 3 \text{ matrices with trace} = 0\}$

trivial subspaces

$W_3 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}$

$W_4 = V$

W_i subspaces of V

If $W \subsetneqq V$ is a subspace, its called a proper subspace of V.

- $W = \{ \text{trace} = 1,\ 3\times 3 \text{ matrices} \}$

$W = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}$

<u>non-examples</u>

<u>Proposition</u>. If $W \subset V$, V is a vector space, then W is a subspace iff W is nonempty and is closed under addition and scalar multiplication.

$$\left.\begin{array}{l} \forall\, u, v \in W: \ u + v \in W \\ \forall\, \alpha \in \mathbb{R},\ \forall u \in W: \ \alpha u \in W. \end{array}\right\} \text{"closure"}$$

Often times, to show $W \neq \phi$ it is easiest to show $\underline{0} \in W$
$\downarrow$
in V

$\boxed{\text{Ex}}$ $W = \{ A \in \mathbb{R}^{2\times 2} \mid tr(A) = 0 \}$ is a subspace of $V = \mathbb{R}^{2\times 2}$. Because

1) $W \neq \phi$, $\underline{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in V$, $tr(\underline{0}) = 0 + 0 = 0$
$\Rightarrow \underline{0} \in W$. ✓

2) Let $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $V = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$, $\underline{U, V \in W}$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$a + d = 0 \qquad\qquad e + h = 0$$

$$U + V = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix} \Rightarrow tr\,(U+V) = (a+e) + (d+h)$$

$$= (a+d) + (e+h)$$

$$= 0 + 0 = 0$$

$$\alpha U = \begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{pmatrix} \Rightarrow tr\,(\alpha U) = \alpha a + \alpha d = \alpha\,(a+d) = \alpha (0).$$

Later, we will see that $tr : \mathbb{R}^{2\times 2} \longrightarrow \mathbb{R}$

is a "linear transformation",

$$\underset{\text{kernel}}{W = \ker}\,(tr) \quad \text{Subspace of } \mathbb{R}^{2\times 2}$$

$\underline{Ex}$ $W = \{ A \in \mathbb{R}^{2\times 2} \mid \det(A) = 0 \} \subset \mathbb{R}^{2\times 2}$

$\underline{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in W$ ✓

$\det(A) = 0$, $\det(\alpha A) = \alpha^2 \det(A) = \alpha^2 \cdot 0$ ✓

But $\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{\in W} + \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{\in W} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \notin W$