Span

Let $S \subset V$ be a subset. The <u>span</u> of S is the set

$$\text{span}(S) := \{ a_1 v_1 + \cdots + a_n v_n \mid \underset{n \in \mathbb{N}}{a_1, \dots, a_n \in \mathbb{R}, v_1, \dots, v_n \in S} \}$$

$$\text{span}(\phi) := \{ \underline{0} \}$$

Note

① $\text{span}(S) \subset V$.

② $S \subset \text{span}(S)$

Prop. If $T \subseteq \text{span}(S)$, then $\text{span}(T) \subseteq \text{span}(S)$.

Proof. "A linear combination of linear combinations is also a linear combination."

Corollary. If $T \subseteq \text{span}(S)$ then $\text{span}(T \overset{\text{union}}{\cup} S) = \text{span}(S)$.

proof. $\text{span}(S \cup T) \supseteq \text{span}(S)$ b/c $S \subseteq S \cup T$

also $\text{span}(S \cup T) \subseteq \text{span}(S)$ b/c $S \cup T \subseteq \text{span}(S)$.

Proposition

If $S \subset V$ then $\text{Span}(S)$ is a subspace of V.

Proof. If $S = \phi \Rightarrow \text{Span}(S) = \{\underline{0}\}$ a subspace of V.

Let $S \neq \phi$. To show $\text{Span}(S)$ is a subspace of V by proving the "closure" properties:

$u_1, u_2 \in \text{Span}(S) \Rightarrow u_1 + u_2$ is a linear comb. of vectors in S $\Rightarrow u_1 + u_2 \in \text{Span}(S)$.

a linear combination of vectors in S

$u \in \text{Span}(S) \Rightarrow u = a_1 v_1 + \cdots + a_n v_n$ $\quad a_j \in \mathbb{R}$, $v_j \in S$

$\alpha \in \mathbb{R}$

$\Rightarrow \alpha u = (\alpha a_1) v_1 + \cdots + (\alpha a_n) v_n \in \text{Span}(S)$

$\square$

Ex The solution set of the equation

$$x + 2y = 0$$

is subspace of $\mathbb{R}^2$:

$x = -2y$ $\quad \underline{y = s}$, $x = -2s$

$$S = \{(-2s, s) \mid s \in \mathbb{R}\} = \{s(-2,1) \mid s \in \mathbb{R}\}$$

$= \text{Span}\{(-2,1)\} \Rightarrow S$ is a subspace of $\mathbb{R}^2$.

Ex. $x + 2y = 1$

$y = s, \quad x = 1 - 2s$

$(1-2s, s) = (1,0) + s(-2,1)$

$S = \{(1,0) + s(-2,1) \mid s \in \mathbb{R}\}$ is not a subspace of $\mathbb{R}^2$

$\underline{0} \notin S \Rightarrow S$ is not a subspace

Definition

A $\boxed{\text{spanning set}}$ of a vector space is a subset $S \subseteq V$ such that $\text{Span}(S) = V$.

Obviously, V is a spanning set for V.

Examples $V = \mathbb{R}^2$

1) $S = \{(1,0), (0,1)\}$

let $(x,y) \in \mathbb{R}^2$ be arbitrary.

$$\boxed{(x,y) = x(1,0) + y(0,1)}$$

$(x,y) = \alpha(1,0) + \beta(0,1)$
$= (\alpha, \beta)$

$$\begin{cases} \alpha = x \\ \beta = y \end{cases}$$

$\Rightarrow S$ is a spanning set.

2) $S = \{(0,1), (\frac{1}{2}, 0)\}$

$$(x,y) = \overset{?}{\alpha}(0,1) + \overset{?}{\beta}(\tfrac{1}{2}, 0) \qquad \begin{cases} \alpha = y \\ \beta = 2x \end{cases}$$

$$= (0,\alpha) + (\beta/2, 0)$$

$$= (\beta/2, \alpha)$$

$(5,-3) = -3(0,1) + 10(1/2, 0)$. $\Rightarrow$ S is a spanning set.

3) $S = \{(1,2), (0,-1)\}$

$$(x,y) = \alpha(1,2) + \beta(0,-1) = (\alpha, 2\alpha - \beta) \qquad \begin{cases} \alpha = x \\ \beta = 2x - y \end{cases}$$

$$(x,y) = x(1,2) + (2x-y)(0,-1).$$

$\Rightarrow$ S is spanning set.

4) $S = \{(1,2), (1,-1)\}$

$$(x,y) = \alpha(1,2) + \beta(1,-1) = (\alpha+\beta, 2\alpha - \beta)$$

$$\begin{cases} \alpha + \beta = x \\ 2\alpha - \beta = y \end{cases} \qquad \begin{aligned} \alpha &= \tfrac{1}{3}(x+y) \\ \beta &= \tfrac{1}{3}(2x - y) \end{aligned}$$

$\Rightarrow$ S is spanning.

5) $S = \{(1,0), (0,1), (\pi, e)\}$

$$(x,y) = x(1,0) + y(0,1) + 0(\pi, e).$$

$$\alpha(1,0) + \beta(0,1) + \gamma(\pi, e)$$

$$= (\alpha + \gamma\pi, \beta + \gamma e)$$

$$\begin{cases} 1\alpha + 0\beta + \pi\gamma = x \\ 0\alpha + 1\beta + e\gamma = y \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & \pi & x \\ 0 & 1 & e & y \end{array}\right)$$

$$\begin{array}{ccc} & & \uparrow \\ & & \text{free column} \end{array}$$

$\alpha = x - \pi\gamma$

$\beta = y - e\gamma$

$$(\alpha, \beta, \gamma) = (x - \pi\gamma,\ y - e\gamma,\ \gamma)$$

$$= (x, y, 0) - \underbrace{\gamma}_{\text{any real number}}(\pi, e, -1)$$

6) $S = V = \mathbb{R}^2$ $\qquad$ $\underline{(x,y) = 1\,(x,y)}$

$\Rightarrow$ V is a spanning set for V.

<u>non examples</u>

1) $S = \{(0,1)\}$

$$\text{Span}(S) = \{s(0,1) \mid s \in \mathbb{R}\}$$
$$= \{(0,s) \mid s \in \mathbb{R}\}$$

for example $(4,3) \notin \text{Span}(S)$.

2) $S = \{(1,2), (3,6), (-1,-2)\}$

$\text{Span}(S) = \text{Span}\{(1,2)\}$ $\quad$ b/c $(3,6), (-1,-2) \in \text{Span}\{(1,2)\}$

$$= \{s(1,2) \mid s \in \mathbb{R}\}$$
$$= \{(s, 2s) \mid s \in \mathbb{R}\}$$

$(0,1) \notin \text{Span}(S)$

3) $S = \phi$, $\operatorname{span}(\phi) = \{\underline{0}\} = \{(0,0)\} \neq \mathbb{R}^2$

4) $S = \{\underline{0}\} = \{(0,0)\} \Rightarrow \operatorname{span}(S) = \{\underline{0}\} \neq \mathbb{R}^2$

Linear Independence

A subset $S \subset V$ is called linearly independent

if any equation

$$\alpha_1 v_1 + \dots + \alpha_n v_n = \underline{0} \qquad \underline{v_1, \dots, v_n \in S}$$

has a unique solution $\underline{\alpha_1 = \dots = \alpha_n = 0}$.

Ex $S = \{(1,0), (0,1)\} \subset \mathbb{R}^2 = V$

$$\alpha(1,0) + \beta(0,1) = (0,0)$$

$$(\alpha, \beta) = (0,0) \Rightarrow \underline{\alpha = \beta = 0}.$$

S is linearly independent.

Ex $S = \{(1,0), (0,1), (0,0)\} \subset V = \mathbb{R}^2$

$$(0,0) = \underset{0}{\alpha}(1,0) + \underset{0}{\beta}(0,1) + \underset{\text{any}}{\gamma}(0,0)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right)$$

free

S is not linearly independent.

$S = \{(1,0), (0,1), (1,1)\}$

$$\underline{1}(1,0) + 1(0,1) + (-1)(1,1) = (0,0)$$

Ex $\mathbb{P}_1 = \{ a_0 + a_1 x \mid a_0, a_1 \in \mathbb{R} \}$: polynomials of degree at most 1.

$S = \{ 2+x,\ 4-5x,\ 1-3x \}$

is linearly dependent

$$1(2+x) - 1(4-5x) + 2(1-3x) = 0 = \underline{0}$$

$$\boxed{\alpha(2+x) + \beta(4-5x) + \gamma(1-3x) = 0}$$

$$(2\alpha + 4\beta + \gamma) + (\alpha - 5\beta - 3\gamma)x = 0$$

$$\begin{cases} 2\alpha + 4\beta + \gamma = 0 \\ \alpha - 5\beta - 3\gamma = 0 \end{cases}$$

$$\left(\begin{array}{ccc|c} 2 & 4 & 1 & 0 \\ 1 & -5 & -3 & 0 \end{array}\right)$$ has infinitely many solutions (α, β, γ) including $(0,0,0)$.

Ex $S = \{u, v\} \subset V$ is linearly independent

S is linearly $\underline{\text{dependent}}$ when we can find

α, β, at least one of them non zero, such that

$$\alpha u + \beta v = \underline{0}$$

if $\beta \neq 0 \Rightarrow v = -\dfrac{\alpha}{\beta} u \Rightarrow v$ is a scalar multiple of u

notation

$\underline{\underline{u \parallel v}}$ if $\{u, v\}$ is linearly dependent.

$\underline{P_1}$:

$1-x \parallel 5-5x$

$1-x \nparallel 5-4x$

Remarks

1) A subset of a linearly independent set is linearly independent.

2) A superset of a linearly dependent set is linearly dependent.

$\underline{Ex}$ Is $S = \{ x = (1,2,3),\ y = (0,3,2),\ z = (-1,1,-1) \} \subset \mathbb{R}^3$

linearly independent?

$\alpha x + \beta y + \gamma z = 0.$

$$\begin{array}{ccc} + & - & + \\ \end{array}$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & 1 \\ 3 & 2 & -1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\underbrace{\begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ x & y & z \end{array}}$$

$\Downarrow$

determinant

$$\begin{vmatrix} 3 & 1 \\ 2 & -1 \end{vmatrix} + (-1) \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = -5 - (4-9) = 0$$

$$\begin{array}{ccc|c} x & y & z & \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 2 & 3 & 1 & 0 \\ 3 & 2 & -1 & 0 \end{array}\right]$$

Gauss elimination (Jordan)

$$\text{RRef} = \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

free

$$\boxed{z = -x + y}$$

what about $S_1 = \{x, y\}$?

$$\alpha x + \beta y = (0,0,0)$$

$$\left[\begin{array}{cc|c} 1 & 6 & 0 \\ 2 & 3 & 0 \\ 3 & 2 & 0 \end{array}\right] \xrightarrow{\text{RRef}} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Basis

A basis for a vector space is a linearly independent spanning subset.

Spanning property : makes a basis 'too large'.

linear independence property : " " " 'too small'

Ex The set $B = \{1+x, 2-x\}$ is a basis for $\mathbb{P}_1$.

1) B is spanning :

$$\underbrace{a + bx}_{\text{any}} \overset{?}{=} \alpha\,(1+x) + \overset{?}{\beta}\,(2-x)$$

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}}_{\Downarrow A} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\det = -1 - 2 = -3 \neq 0 \Rightarrow \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = A^{-1} \begin{bmatrix} a \\ b \end{bmatrix}.$$

$\Rightarrow$ B is spanning

2) B is linearly independent.

$$0 = \alpha(1+x) + \beta(2-x)$$

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\det \neq 0 \Rightarrow \alpha = \beta = 0$