Lecture 2

* Matrices of a system of Equations

Definition $m \times n$ matrix

m ↓ # rows, n ↘ # columns

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

$a_{ij} \in \mathbb{R}$ (or $\mathbb{C}$)

$\{$all $m \times n$ matrices$\}$: $\mathbb{R}^{m\times n}$, $M_{mn}(\mathbb{R})$

Given a system

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ \vdots \qquad\qquad & \quad \vdots \\ a_{m1}x_1 + \cdots\ a_{mn}x_n &= b_m \end{aligned}$$

Coefficient matrix $= \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \in \mathbb{R}^{m\times n}$

variable vector $= \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^{n\times 1}$

constant vecter $= \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \in \mathbb{R}^{m\times 1}$

Augmented matrix $\left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & & \vdots \\ a_{m1} & & a_{mn} & b_m \end{array}\right] \in \mathbb{R}^{m\times(n+1)}$

$$y + 3x = -1$$

Ex

$$x = 2$$

variable vector $= \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^{2\times 1}$ Constant vector $= \begin{bmatrix} -1 \\ 2 \end{bmatrix} \in \mathbb{R}^{2\times 1}$

coefficient matrix $= \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix} \in \mathbb{R}^{2\times 2}$

augmented matrix $= \left[\begin{array}{cc|c} 3 & 1 & -1 \\ 1 & 0 & 2 \end{array}\right] \in \mathbb{R}^{2 \times 3}$

Homogeneous : constant vector = null = $\vec{0}$

Definition **Elementary Row operations**

1) Switching two rows : $R_i \leftrightarrow R_j$ $i \neq j$

2) multiplying a row by a non zero scalar

"replaced with"

$R_i \leftarrow c R_i$ $\underline{c \neq 0}$

read from left to right

3) adding a scalar multiple of a row to another row

$R_i \leftarrow R_i + c R_j$

Definition Two matrices are row equivalent if one can be gotten from the other using a few elementary row operations.

Corollary: Equivalent systems means row equivalent augmented matrices.

Example
$$\left[\begin{array}{cc|c} 3 & 1 & -1 \\ 1 & 0 & 2 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 3 & 1 & -1 \end{array}\right]$$

$$\xrightarrow{R_2 \leftarrow R_2 - 3R_1} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -7 \end{array}\right]$$

$$\Updownarrow$$

$$1x + 0y = 2 \qquad \left\{ \begin{array}{l} x = 2 \\ y = -7 \end{array} \right.$$
$$0x + 1y = -7$$

$$S = \{(2,-7)\} \subset \mathbb{R}^2$$

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* Back substitution

$$\left[\begin{array}{cccc|c} 2 & 1 & 0 & -1 & 4 \\ 0 & 0 & 3 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

\# variables = 4

$x_1, \dots, x_4$

$$2x_1 + x_2 + \quad - x_4 = 4$$
$$3x_3 + x_4 = 2 \implies x_3 = 2/3 - x_4/3$$
$$0 = 0$$

x_1, x_3: leading variables

free: x_2 and x_4

$x_2 \to t$, $x_4 \to s$

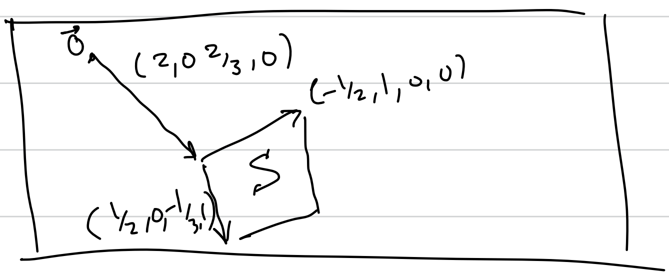
$$\left\lfloor x_3 = \tfrac{2}{3} - \tfrac{s}{3} \quad , \quad x \right.$$

$$\left\lfloor 2x_1 = -x_2 + x_4 + 4 = -t + s + 4 \right.$$

$$\boxed{x_1 = -\tfrac{t}{2} + \tfrac{s}{2} + 2}$$

$$S = \left\{ \left(-\tfrac{t}{2} + \tfrac{s}{2} + 2,\ t,\ \tfrac{2}{3} - \tfrac{s}{3},\ s\right) \;\middle|\; s, t \in \mathbb{R} \right\}$$

$$= \left\{ (2, 0, \tfrac{2}{3}, 0) + s\left(\tfrac{1}{2}, 0, -\tfrac{1}{3}, 1\right) + t\left(-\tfrac{1}{2}, 1, 0, 0\right) \;\middle|\; s, t \in \mathbb{R} \right\}$$



<u>Def'n</u> A matrix is in **row echelon form** if the number of <u>leading zeros</u> in the rows is strictly increasing from top to bottom until we have (possibly) <u>zero rows</u>.

<u>Definition</u> Let A be a matrix in row echelon form. The first nonzero entry in each row of A is called a <u>leading entry</u>. The columns of A corresponding

to the leading entries are called <u>leading columns</u>.

The remaining columns = free columns.

<u>Ex</u>

leading 0: 0, 1, hits a zero row. $\Rightarrow$ <u>echelon form</u>

$$\begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

leading columns, no free.

$$\begin{bmatrix} 0 & 3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ echelon ✓

free, leading, free

free

$$\begin{bmatrix} 2 & -1 & 2 & 0 \\ 0 & 0 & 3 & 1 \end{bmatrix}$$

echelon ✓

leading columns

free, free, free

$$\begin{bmatrix} 0 & 2 & 0 & 1 \end{bmatrix}$$

leading column

$$\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$$ not row echelon form

leading 0's: 1, 1 not strictly increasing.

$$\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$$ not row echelon: <u>1, 0</u>

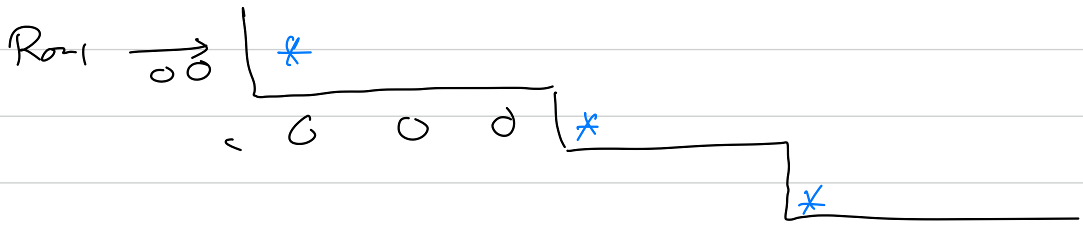
<u>Exercise</u>: Find all 2x2 matrices in row echelon form.

<u>Gauss elimination</u>

$$A \xrightarrow{\text{elem. row operations}} \text{An echelon form}$$

<u>Idea</u>: Pick the first row that corresponds to

a leading entry and subtract an appropriate multiple of this row from the following rows.



Ex
$$\begin{bmatrix} 0 & -2 & 0 & 2 \\ 2 & 4 & 8 & 2 \\ 3 & 1 & -1 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 2 & 4 & 8 & 2 \\ 0 & -2 & 0 & 2 \\ 3 & 1 & -1 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 \leftarrow \frac{1}{2} R_1} \begin{bmatrix} 1 & 2 & 4 & 1 \\ 0 & -2 & 0 & 2 \\ 3 & 1 & -1 & 0 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 - 3R_1} \begin{bmatrix} 1 & 2 & 4 & 1 \\ 0 & -2 & 0 & 2 \\ 0 & -5 & -13 & -3 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftarrow \frac{-1}{2} R_2} \begin{bmatrix} 1 & 2 & 4 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & -5 & -13 & -3 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 + 5R_2} \begin{bmatrix} 1 & 2 & 4 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -13 & -8 \end{bmatrix}$$

Note: a matrix can be row equivalent to several matrices in row echelon form.

Solving Systems of equations

Algorithm.

1) Find the augmented matrix.
2) Use Gauss elimination to find an echelon form of the augmented matrix.

3) There is no solution (inconsistent) if the augmented column of the echelon form is leading.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

A leading column

4) Identifying the free variables (if any) corresponding to free columns.

5) Set the free variables equal to parameters.

6) Use back substitution to find the leading variables.

7) Build the solution set.

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & 4 & 3 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 0 & 1 \end{array}\right]$$ is inconsistent.

Gauss-Jordan reduction

Reduced Row Echelon form:

Row Echelon + $\begin{cases} \text{1) Every leading entry is } 1. \\ \text{2) every leading entry is the only nonzero entry in its column} \end{cases}$

$$\begin{bmatrix}0\\0\\0\\1\\0\\0\\0\end{bmatrix}$$

$\downarrow$

a leading column

$$\begin{array}{c}\begin{array}{cccc}\checkmark & \checkmark & & \end{array}\end{array}$$

$$\begin{bmatrix}1 & 2 & 4 & 1\\ 0 & 1 & 0 & -1\\ 0 & 0 & -13 & -8\end{bmatrix} \xrightarrow{R_1 \leftarrow R_1 - 2R_2} \begin{bmatrix}\underline{1} & 0 & 4 & 3\\ 0 & \underline{1} & 0 & -1\\ 0 & 0 & \underline{\underline{-13}} & -8\end{bmatrix}$$

$$\xrightarrow{R_3 \leftarrow \frac{R_3}{-13}} \begin{bmatrix}1 & 0 & 4 & 3\\ 0 & 1 & 0 & -1\\ 0 & 0 & 1 & \frac{8}{13}\end{bmatrix}$$

$$\xrightarrow{R_1 \leftarrow R_1 - 4R_3} \begin{bmatrix}1 & 0 & 0 & 3-\frac{32}{13}\\ 0 & 1 & 0 & -1\\ 0 & 0 & 1 & 8/13\end{bmatrix} \quad \underline{\underline{\text{RREF}}} \checkmark$$

Theorem. Every matrix A is row equivalent to a unique <u>reduced row echelon form</u> RREF.

$\boxed{2}$ Matrices

$\mathbb{R}^{n\times 1} \equiv$ column matrices $\equiv \mathbb{R}^n$

$\mathbb{R}^{1\times n} \equiv$ row matrices

$\mathbb{R}^{1\times 1} \equiv \mathbb{R}$ $\qquad [3] \equiv 3$

$$\begin{bmatrix}1\\2\end{bmatrix} \in \mathbb{R}^{2\times 1} \qquad\qquad [1 \quad 2]$$

$$\equiv \qquad\qquad\qquad\qquad \not\equiv$$

$$(1,2) \in \mathbb{R}^2 \qquad\qquad (1,2)$$

- j-th unit vector $e_j \in \mathbb{R}^n$

$$(0, 0, \dots, \underset{\substack{\uparrow \\ j\text{-th}}}{1}, 0, 0, 0, \dots) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \rightarrow j\text{-th row}$$

- Squar matrix $\underline{m = n}$.

- Diagonal : $m = n$ (square) + $a_{ij} = A(i,j) = 0$ $\underline{\text{for } i \neq j}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{diagonal}$$

* $I_n = [e_1 \ e_2 \ \cdots \ e_n] = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \vdots & & \vdots \\ \vdots & \vdots & \vdots & & 1 \end{bmatrix}$