Given an ordered basis

$$E = (b_1, b_2, \cdots, b_n)$$

for $V$, each $v \in V$ can be written

<u>uniquely</u> as

$$v = \alpha_1 b_1 + \alpha_2 b_2 + \cdots + \alpha_n b_n \qquad (\alpha_j \in \mathbb{R})$$

$$[v]_E = (\alpha_1, \cdots, \alpha_n) \in \mathbb{R}^n$$

Coordinate of $v$ with respect to $E$.

Notice the function

$$\begin{cases} V \longrightarrow \mathbb{R}^n \\ v \longmapsto [v]_E \end{cases}$$

is a bijection : one to one and onto.

<u>one to one</u>, $\quad v \neq w \implies [v]_E \neq [w]_E$

$(b_1, \cdots, b_n)$

If $[v]_E = [w]_E = (\alpha_1, \cdots, \alpha_n)$

$$v = \sum_{j=1}^{n} \alpha_j b_j = w$$

<u>onto</u>: Given $(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$, then

$$[\underbrace{\sum_{j=1}^{n} \alpha_j v_j}_{\Downarrow \atop \in V}] = (\alpha_1, \ldots, \alpha_n)$$

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Moreover $v \longmapsto [v]_E$, is "<u>linear</u>":

- $[u+v]_E = [u]_E + [v]_E$
- $[ku]_E = k[u]_E$

let

$$[u]_E = (\alpha_1, \ldots, \alpha_n) \qquad [v]_E = (\beta_1, \ldots, \beta_n)$$

$$u = \sum_1^n \alpha_j b_j \quad , \quad v = \sum_1^n \beta_j v_j$$

$$u+v = \sum_1^n \underline{(\alpha_j + \beta_j)} v_j \Rightarrow [u+v]_E = (\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n)$$

$$= (\alpha_1, \ldots, \alpha_n) + (\beta_1, \ldots, \beta_n)$$

$$ku = k \sum_1^n \alpha_j b_j = \sum_1^n \underline{(k\alpha_j)} b_j \Rightarrow [ku]_E = (k\alpha_1, \ldots, k\alpha_n)$$

$$= k(\alpha_1, \ldots, \alpha_n)$$

$\Rightarrow$ Indeed, $[v] \longrightarrow [v]_E$, $V \rightarrow \mathbb{R}^n$ is an <u>isomorphism</u>

$\Rightarrow V \cong \mathbb{R}^n$ : $V$, $\mathbb{R}^n$ are essentially the same as vector spaces.

**Ex** $\quad E = (\overset{(1,0)}{e_1}, \overset{(0,1)}{e_2})$ in $\mathbb{R}^2$.

$\Rightarrow \quad [(a,b)]_E = (a,b) \qquad\qquad (a,b) = ae_1 + be_2$

**Ex** $\quad F = (e_2, e_1)$ in $\mathbb{R}^2$

$[(a,b)]_F = (b,a)$

**Ex** $\quad \mathbb{P}_1[x] = \{ a+bx \mid a,b \in \mathbb{R} \}$

standard basis: $(1, x) = E$

$[-1-8x]_E = (-1, -8)$

Another basis: $\quad B = (1-x, 1+2x)$

$[-1-8x]_B = (\alpha, \beta) \in \mathbb{R}^2$

meaning $\quad -1-8x = \alpha(1-x) + \beta(1+2x)$

$= (\alpha+\beta) + (-\alpha+2\beta)x$

$\Rightarrow \begin{cases} \alpha+\beta = -1 \\ -\alpha+2\beta = -8 \end{cases} \qquad 3\beta = -9 \Rightarrow \beta = -3$

$\alpha = 2$

$[-1-8x]_B = (2, -3)$

# Example for applying Coordinates

**Algorithm**: To find a basis for the subspace

$$W = \operatorname{span}\{v_1, \ldots, v_n\}$$

of $V$.

① Find a basis for $V$, call it $B$.
use a standard one, if possible

② Build a matrix $A$ using

$$[v_1]_B, \ldots, [v_n]_B$$

as columns.

③ Row reduce $A$ to an echelon form $R$.

④ Take the columns of $A$ corresponding to the leading columns of $R$.

⑤ Use these columns as coordinate vectors to build a basis for $W$.

Ex Find a basis for

$$W = \text{Span}\{1-x,\ 2+x^2,\ 5-x+2x^2\} \subseteq \mathbb{P}_2[x]$$

$\mathbb{P}_2[x]$: dim = 3

$E = (1, x, x^2)$

$$A = \begin{bmatrix} 1 & 2 & 5 \\ -1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} = R$$

leading

$\Rightarrow (1,-1,0),\ (2,0,1)$ form a basis for the Col(A).

$\Rightarrow \{1-x,\ 2+x^2\}$ form a basis for $W$.

$\Rightarrow (\dim W = 2)$
$= \text{rank} A = \text{rank}\, R$

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## Change of basis

Example $V = \mathbb{R}^2$. $E = ((1,0), (0,1))$

$B = ((2,0), (1,1))$ $\quad \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} \neq 0$

let $w \in V$ such that $[w]_B = (3,-2)$.

$\Rightarrow \quad w = 3(2,0) - 2(1,1) = (4,-2)$

$\Rightarrow \quad [w]_E = (4,-2)$.

Note that

$$[w]_E = \begin{bmatrix} 4 \\ -2 \end{bmatrix} = 3\begin{bmatrix} 2 \\ 0 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

$$[w]_E = \underbrace{\left[ (2,0)_E \;\vdots\; (1,1)_E \right]}_{\text{transition matrix } T_B^E \text{ from } B \text{ to } E} [w]_B$$

$$\boxed{[w]_E = T_B^E [w]_B}$$

Definition Let $B = (b_1, \dots, b_n)$ and $C = (c_1, \dots, c_n)$ be two ordered bases for $V$. Then transition matrix from $B$ to $C$ is defined by

$$T_B^C = \left[ [b_1]_C \cdots [b_n]_C \right]$$

Proposition $[v]_C = T_B^C [v]_B$

Proof.

$$B = (b_1, \dots, b_n)$$

$$[v]_B = (\beta_1, \dots, \beta_n)$$

$$\Rightarrow v = \beta_1 b_1 + \dots + \beta_n b_n$$

$$[v]_C = [\beta_1 b_1 + \dots + \beta_n b_n]_C = \beta_1 [b_1]_C + \dots + \beta_n [b_n]_C$$

$$= \left[ [b_1]_C \cdots [b_n]_C \right] \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix} = T_B{}^C [v]_B . \qquad \blacksquare$$

Proposition

$$T_B{}^C = \left( T_C{}^B \right)^{-1}.$$

$$[v]_B = T_C{}^B [v]_C = T_C{}^B \cdot \left( T_B{}^C [v]_B \right) = \underline{T_C{}^B T_B{}^C} [v]_B$$

$$[v]_C = T_B{}^C [v]_B = T_B{}^C \left( T_C{}^B [v]_C \right) = \underline{T_B{}^C T_C{}^B} [v]_C$$

$$T_C{}^B T_B{}^C = I = T_B{}^C T_C{}^B .$$

Ex In a previous example, we found

$$T_B{}^E = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$$

for V with

$E = (e_1, e_2)$

$B = \left\{ \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$

$$T_E{}^B = \left(T_B{}^E\right)^{-1} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}^{-1} = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1/2 & -1/2 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow [(1,2)]_B = T_E^{B} \cdot [(1,2)]_E = \begin{pmatrix} 1/2 & -1/2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 2 \end{pmatrix}$$

$$\Rightarrow [(1,2)]_B = (-1/2, 2)$$

<u>Exercise</u> Find $[2-x]_B$ if $B = (\underbrace{1-x}_{b_1}, \underbrace{1+2x}_{b_2})$

① direct way: $2-x = \alpha(1-x) + \beta(1+2x)$

$$\Rightarrow [2-x]_B = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

② <u>Transition</u> Find $T_E^{B} = \left(T_B^{E}\right)^{-1} = \left([b_1]_E \quad [b_2]_E\right)^{-1}$

$$= \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$$

$$[2-x]_B = T_E^{B}\, [2-x]_E = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 5 \\ 1 \end{pmatrix} = (\underset{\alpha}{\underset{\downarrow}{5/3}}, \underset{\beta}{\underset{\downarrow}{1/3}})$$

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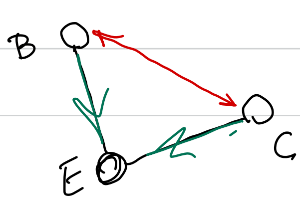
E: a standard basis in V

B, C: two other ordered bases for V




$$T_B^{C} = \left(T_C^{E}\right)^{-1} T_B^{E}$$

$$[v]_C = T_E^C \underbrace{[v]_E} = T_E^C \left(T_B^E [v]_B\right)$$

$$= \underbrace{\left(T_E^C T_B^E\right)} [v]_B$$

$$\Downarrow$$

$$T_B^C = \underbrace{T_E^C} \cdot T_B^E = \left(T_C^E\right)^{-1} T_B^E .$$

Example

$V = \mathbb{R}^2 \xrightarrow{\text{basis}} C = \left((-1,1),(1,2)\right).$

$V = \mathbb{R}^2 \xrightarrow{\text{basis}} B = \left((1,2),(3,4)\right)$

Suppose $[(a,b)]_B = (5,4)$. Find $[(a,b)]_C$.

$$[(a,b)]_C = T_B^C [(a,b)]_B = T_B^C \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

$$= T_E^C T_B^E \begin{bmatrix} 5 \\ 4 \end{bmatrix} = \left(T_C^E\right)^{-1} \cdot T_B^E \cdot \begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 5 \\ 4 \end{pmatrix}$$